Master Thesis

Asymptotic Analysis Methods for Reduced Cardiovascular Blood Flow Modelling

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Abstract

We mainly study the unsteady axial symmetry flow of a Newtonian incompressible fluid in a thin right cylinder whose radius is small with respect to its length. The flow is driven by a given time-dependent pressure drop between the inlet and the outlet boundary. The pressure drop is assumed to be small, therefore introducing creeping flow in the compliant tube. We use Stokes equation to model the fluid and use the Navier equations for the curved, linearly elastic membrane to model the wall. Due to the creeping flow and to small displacements, the interface between the fluid and the lateral wall is linearized and supposed to be the initial configuration of the membrane. We study the dynamics of this coupled fluid-structure system in the limit situation when the ratio between the radius and the length of the tube tends to be zero. Using the asymptotic techniques, we get effective equations (reduced equation).

The applications of this model problem include blood flow in small arteries.

Contents

Chapter 1

Introduction

In this thesis, we study blood flow through compliant vessels. This is a fluid-structure interaction problem between the incompressible Navier-Stokes equation and the motion of a compliant vessel wall. Modelling of the compliant vessel wall is a complex problem in its own right. Even in the simplified case when the anisotropic behavior of the vessel wall is ignored and angular deformations are neglected, in which case the linear Navier equations for the curved membrane can be used to model the wall, the analysis of the nonlinear coupling between the flow equations (Navier-Stokes equations) and wall behavior (Navier membrane equations) is unsolved. We focus on understanding the coupling between the Stokes equations (creeping flow) and the Navier equations for a curved elastic membrane. This is a good model for the blood flow in small arteries. Indeed, it was noted in reference [4] that in small arteries, viscous effects of blood become more important than the inertia effects, and normally in small arteries and capillaries, the convective term, the nonlinear term in Navier-Stokes equations is ignored and therefore Stokes equations without inertia term are more appropriate.

1.1 The Physical Problem

Arteries can be regarded as hollow tubes with strongly variable diameters and can be subdivided into *large arteries, medium arteries, arterioles* and *capillaries.* The main role

type	diameter
large arteries	$1-3cm$
medium arteries	—
arterioles	—
capillaries	$4-10\mu m$

Table 1.1: the diameter of arteries

of large arteries is to carry a substantial blood flow rate from the heart to the periphery and to act as a 'compliant system'. They deform under blood pressure and by doing so they are capable of storing elastic energy during the systolic phase and return it during the diastolic phase. As a result the blood flow is more regular than it would be if the large arteries were rigid. Then we have a *fluid-structure* interaction problem. The blood may be considered a homogeneous fluid, with 'standard' behaviour (Newtonian fluid), the wall may be considered elastic (or mildly visco-elastic).

Indeed, the blood is not a fluid but a suspension of particles in a fluid called *plasma*. Blood particles must be taken into account in the rheological model in smaller arterioles and capillaries since their size becomes comparable to that of the vessel. The most important blood particles are : *red cells*(*erythrocytes*), *white cells*(*leukocytes*) and *platelets*(*thrombocytes*).

1.2 Some Terminologies

Axial symmetry All quantities are independent from the angular coordinate θ. As a consequence, every axial section z=const remains circular during the wall motion.

Newtonian fluid A fluid that has a constant viscosity at all shear rates at a constant temperature and pressure, and can be described by a one-parameter rheological model.

non-Newtonian Viscosity is not a constant and varies with strain rate.

Incompressible fluid A fluid in which the density remains constant for isothermal pressure changes, that is, for which the coefficient of compressibility is zero.

no-slip Velocity of the fluid on the boundary is assumed to be zero.

steady flow means the time derivative of the velocity of the fluid is zero.

1.3 Blood Rheology

The branch of science which studies the behavior of a moving fluid and in particular the relation between stresses and the kinematic quantities is called *rheology*.

Human blood is a suspension of cells in an aqueous solution of electrolytes and non-electrolytes. By centrifugation, the blood is separated into *plasma* and *cells*. The plasma is about 90% water by weight, 7% plasma protein, 1% inorganic substances and 1% other organic substances. The cellular contents are essentially all *erythrocytes* or *red cells* with *white cells* of various categories making up less than $1/600th$ of the total volume and *platelets* less than $1/800th$ of the cellular volume. Normally, the red cells occupy about 50% of the blood volume.

When plasma was tested in a viscometer, it was founded to behave like a Newtonian viscous fluid (Merrill et al.,1965) with a coefficent of viscosity about $1.2cp$ (Gregersen et al., 1967; Chien et al., 1966, 1971). But when the whole blood was tested in a viscometer, its non-Newtonian character was revealed.

The major features of blood flow in a viscometer are of *Couette-flow type.* In large blood vessels, the blood is considered as a *homogeneous fluid.* But when blood flows in capillary blood vessels, the red cells have to be squeezed and deformed and move in single file. In this case, it would be more useful to consider blood as a *nonhomogeneous fluid.*

For biomechanical studies, the constitutive relations are crucial .

In order to provide a brief acquaintance with the complex field of blood rheology, we introduce some basic notations from fluid mechanics. Assume $\mathbf{T}$ denotes *stress tensor* of the fluid and $\mathbf{D}$ denotes *strain rate tensor*, defined as follows:

$$\mathbf{D} = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T)$$

which is symmetric. Assessing the dependence law of $\mathbf{T}$ from $\mathbf{D}$ is the field of *rheology.* This relation is called the *constitutive law* and in many cases, it can be expressed by the equation in the following form:

$$\mathbf{T} = -P\mathbf{I} + \mathbf{S}$$

where $\mathbf{I}$ is the Kronecker tensor (identified by an identity matrix). In this case, tensor P$\mathbf{I}$ is called the *isotropic tensor*, P is the *pressure* of the fluid, while $\mathbf{S}$ is the so called *extra-stress tensor.*

If $\mathbf{S}$ is a linear function of the rate-of-strain tensor, i.e.

$$\mathbf{S} = 2\mu\mathbf{D} = \mu(\nabla \mathbf{u} + \nabla \mathbf{u}^T) \tag{1.1}$$

the fluid is called *Newtonian.* The constant μ represents the *(dynamic) viscosity* of the fluid. The Newtonian law (1.1) is the simplest one which can be encountered in the study of viscous flows.

A very important application of blood rheology in clinical medicine is to identify diseases from any change in blood viscosity.

1.4 Physical meaning of various terms in the equations for the membrane

Inertia term $\rho_\omega h(\varepsilon)\frac{\partial^2 \eta^\varepsilon}{\partial t^2}$ **and** $\rho_\omega h(\varepsilon)\frac{\partial^2 s^\varepsilon}{\partial t^2}$ is proportional to the acceleration of the vessel-wall

term $h(\varepsilon)G(\varepsilon)k(\varepsilon)\frac{\partial^2\eta^\varepsilon}{\partial z^2}$ **and** $\frac{h(\varepsilon)E(\varepsilon)}{1-\sigma^2}\frac{\partial^2 s^\varepsilon}{\partial z^2}$ is related to the longitudinal pre-stress state of the vessl. It is indeed well known that in physiological conditions an artery is subjected to a longitudinal tension.

term $\frac{h(\varepsilon)E(\varepsilon)}{1-\sigma^2}\frac{\eta^\varepsilon}{\varepsilon^2 R^2}$ is the elastic-response function

1.5 Something about Navier-Stokes equations

We consider the Incompressible Navier-Stokes equations

$$\rho\frac{\partial \mathbf{u}}{\partial t} + \rho(\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla\mathbf{p} - 2\nabla\cdot(\mu\mathbf{D}(\mathbf{u})) = \rho\mathbf{f}$$
$$\nabla\cdot\mathbf{u} = 0$$

By introducing the kinematic viscosity $\nu = \frac{\mu}{\rho}$, then we can write above equation in the form

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \frac{1}{\rho}\nabla\mathbf{p} - 2\nabla\cdot(\nu\mathbf{D}(\mathbf{u})) = \mathbf{f}$$
$$\nabla\cdot\mathbf{u} = 0$$

$\frac{\partial \mathbf{u}}{\partial t}$ is the acceleration term, $\nabla\cdot(\nu\mathbf{D}(\mathbf{u}))$ is the viscous term, $\mathbf{f}$ is the external forces, $(\mathbf{u}\cdot\nabla)\mathbf{u}$ is the nonlinear convective term, if the fluid is highly viscous, the contribution of this nonlinear convective term may be neglected. The key parameter which allows us to make this decision is the *Reynolds number Re* which is a non-dimensional number defined as

$$Re = \frac{|\mathbf{u}|L}{\nu}$$

where L represents a characteristic length-scale for the problem at hand and $|\mathbf{u}|$ is the Euclidean norm of the velocity. For the flow in a tube L is the tube diameter. Nevertheless in the situation where $Re \ll 1$ (for instance, flow in smaller arteries or capillaries) we may say that the convective term is negligible compared to the viscous contribution and may be discarded. Then we have the unsteady Stokes equations,

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{1}{\rho}\nabla\mathbf{p} - 2\nabla\cdot(\nu\mathbf{D}(\mathbf{u})) = \mathbf{f}$$
$$\nabla\cdot\mathbf{u} = 0$$

Here, in this thesis, we consider flow in smaller arteries, so Stokes equation is more appropriate to model the fluid, since we have mentioned that in small arteries, the viscous effect is much more important than the inertia effect. So Stokes equation without inertia term is more appropriate here which is as same as the study of the steady flow. If we use the complete Stokes equations to model the fluid, normally to model the structure, we will use the *independent ring model.*

1.6 An introduction of the thesis

For a nice introduction to human cardiovascular blood flow modelling we refer to '*Mathmatical modelling and numerical simulation of the cardiovascular system*' written by Alfio Quarteroni and Luca Formaggia. A mathematical and computational challenge in the modelling of fluid flow in an elastic membrane is the coupling of fluid and elastic equations. One way to simplify the situation is to use asymptotic analysis and try to derive asymptotics equations (hopefully reduced equations). These reduced equations might be different for different size of the vessels.

In this thesis, we follow the work by Čanić and Mikelić and study the asymptotic equations of small (capillary size) vessels. Starting from the coupled Navier equations for the elastic membrane and the axisymmetric Stokes equation for the flow we derive rigorously one asymptotic equation, from this equation, we can get a parabolic equation for the pressure when the shear modulus is neglected or zero and a fourth order equation when the shear modulus is not neglected.

For the two equations, We perform numerical simulations using $CG1$ and $CG2$ respectly. We also study the full system numerically by using **Comsol Multiphysics**.

The thesis is organized as follows:

In section 2, we present the problem and introduce the coupled fluid-structure equations both in strong and variational form. At the end of this section, we give a sketch of the proof of the existence and uniqueness of the solution to **FSI** problems. In section 3 we prove all the technical a priori estimates. In section 4 , we introduce the rescaled problem and state and prove the a priori estimates for the rescaled quantities and introduce the appropriate asymptotic expansions. In section 5, we use asymptotic techniques to derive the reduced problem, a heat equation for the pressure, indeed there are two possibilities for the reduced equation. For small and negligible shear modulus , the asymptotic equation is a standard heat equation. For nonnegligible shear modulus, the asymptotic equation is a fourth order equation. In section 6, we collect the convergence results in the main **Theorem 4**, where we prove that the rescaled system converges to the reduced equation. In section 7, we present a **Comsol Multiphysics** simulation of our blood flow in elastic membrane model in 3D. In section 8, appendix, finally we collect some mathematical results and notations which are needed in this work.

#	Artery	Length(cm)	Area(cm^2)	$\beta(kg \cdot s^{-2} \cdot cm^2)$	R_t
1	Ascending Aorta	4.0	5.983	97	-
2	Aortic Arch I	2.0	5.147	87	-
3	Brachiocephalic	3.4	1.219	233	-
4	R.Subclacian I	3.4	0.562	423	-
5	R.Carotid	17.7	0.432	516	-
6	R.Vertebral	14.8	0.123	2590	0.906
7	R.Subclavian II	42.2	0.510	466	-
8	R.Radial	23.5	0.106	2866	0.82
9	R.Ulnar I	6.7	0.145	2246	-
10	R.Interosseous	7.9	0.031	12894	0.956
11	R.Ulnar II	17.1	0.133	2446	0.893
12	R.Internal Carotid	17.6	0.121	2644	0.784
13	R.External Carotid	17.7	0.121	2467	0.79
14	Aortic Arch II	3.9	3.142	130	-
15	L.Carotid	20.8	0.430	519	-
16	l.Internal Carotid	17.6	0.121	2644	0.784
17	l.External Carotid	17.7	0.121	2467	0.791
18	Thoracic Aorta I	5.2	3.142	124	-
19	L.Subclacian I	3.4	0.562	416	-
20	Vertebral	14.8	0.123	2590	0.906
21	L.Subclacian II	42.2	0.510	466	-
22	L.Radial	23.5	0.106	2866	0.821
23	L.Ulnar I	6.7	0.145	2266	-
24	L.Interosseous	7.9	0.031	12894	0.956
25	L.Ulnar II	17.1	0.133	2446	0.893
26	Intercostals	8.0	0.196	885	0.627
27	Thoracic Aorta II	10.4	3.017	117	-
28	Abdominal I	5.3	1.911	167	-
29	Celiac I	2.0	0.478	475	-
30	Celiac II	1.0	0.126	1805	-
31	Hepatic	6.6	0.152	1142	0.925
32	Gastric	7.1	0.102	1567	0.921
33	Splenic	6.3	0.238	806	0.93
34	Superior Mesenteric	5.9	0.430	569	0.934
35	Abdominal II	1.0	1.247	227	-
36	L.Renal	3.2	0.332	566	0.861
37	Abdominal III	1.0	1.021	278	-
38	R.Renal	3.2	0.159	1181	0.861
39	Abdominal IV	10.6	0.697	381	-
40	Inferior Mesenteric	5.0	0.080	1895	0.918
41	Abdominal V	1.0	0.578	399	-
42	R.Common Iliac	5.9	0.328	649	-
43	L.Common Iliac	5.8	0.328	649	-
44	L.External Iliac	14.4	0.252	1493	-
45	L.Internal Iliac	5.0	0.181	3134	0.925
46	L.Femoral	44.3	0.139	2559	-
47	L.Deep Femoral	12.6	0.126	2652	0.885
48	L.Posterior Tibial	32.1	0.110	5808	0.724
49	L.Anterior Tibial	34.3	0.060	9243	0.716
50	R.External Iliac	14.5	0.252	1493	-
51	R.Internal Iliac	5.1	0.181	3134	0.925
52	R.Femoral	44.4	0.139	2559	-
53	R.Deep Femoral	12.7	0.126	2652	0.888
54	L.Posterior Tibial	32.2	0.110	5808	0.724
55	R.Anterior Tibial	34.4	0.060	9243	0.716

Table 1.2: the values of different parameters of 55 main arteries in the human arterial system

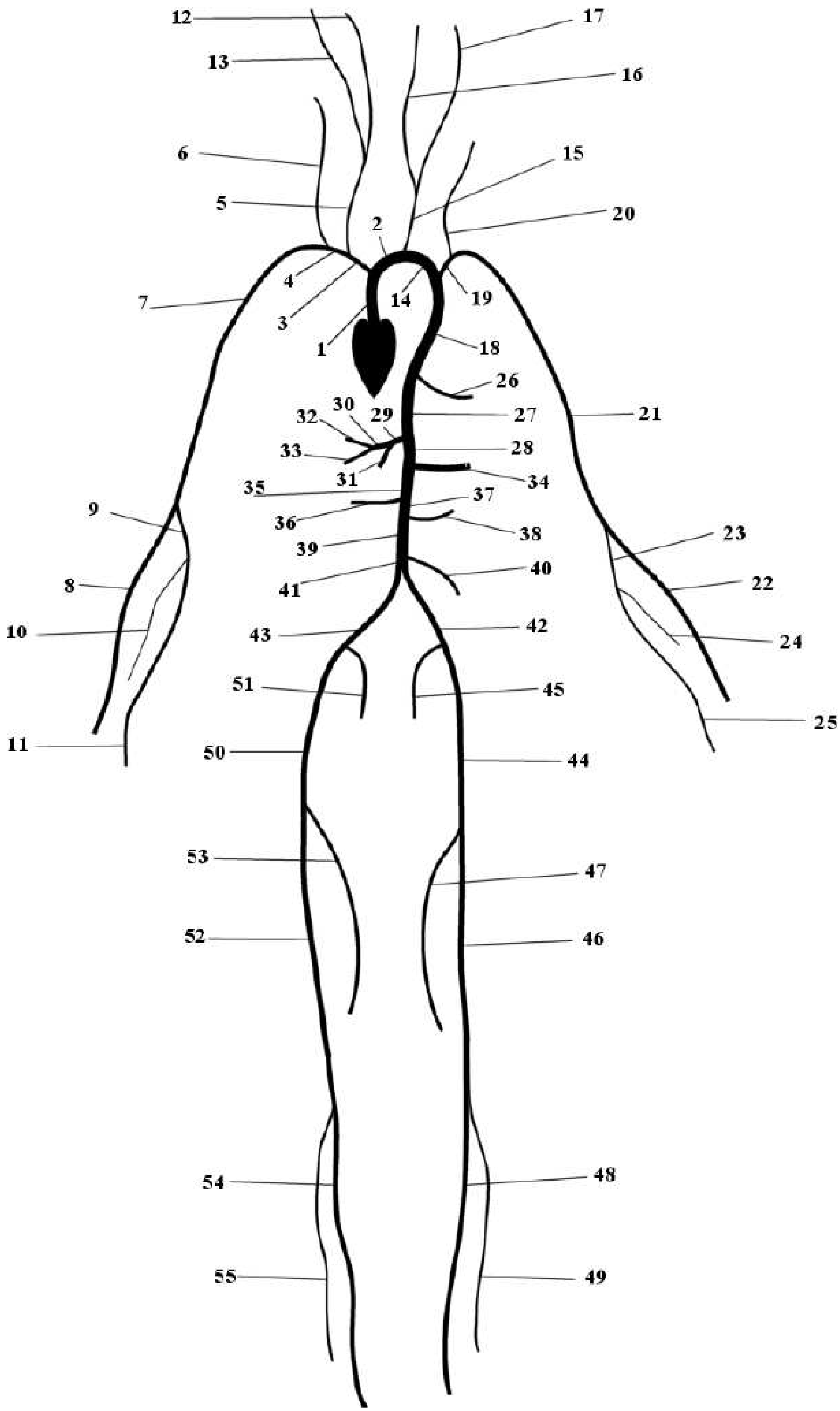


Figure 1.1: Connectivity of the 55 main arteries in the human arterial system.@reproduced by permission of Luca Formaggia

Chapter 2

Presentation of the problem

2.1 Presentation of the problem

We consider the unsteady axisymmetric flow of a Newtonian incompressible fluid in a thin right cylinder whose radius is small with respect to its length, $\varepsilon = \frac{r}{L}$, where r is the radius and L is the length of the cylinder. For each fixed $\varepsilon > 0$, introduce Ω_ε to be

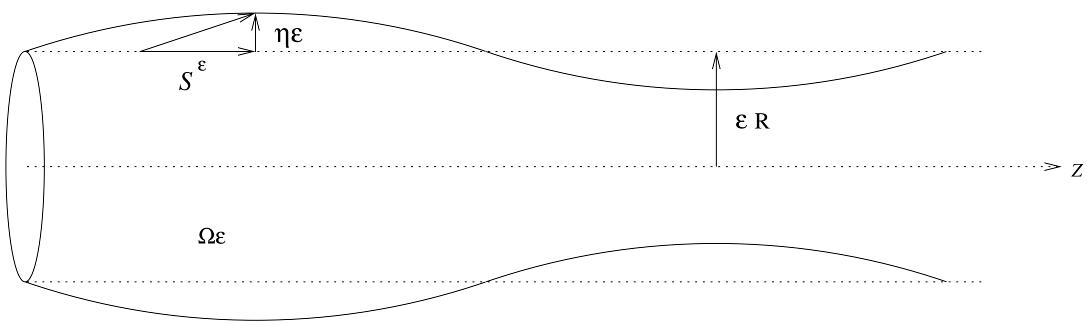


Figure 2.1: Wall displacement

$$\Omega_\varepsilon = \{x \in \mathbb{R}^3; x = (r\cos\vartheta, r\sin\vartheta, z), r < \varepsilon R, 0 < z < L\}. \tag{2.1}$$

We assume that the cylinder's lateral wall $\Sigma_\varepsilon = \{r = \varepsilon R\} \times (0, L)$ is elastic and that its motion is described in Lagrangian coordinates by the Navier equations

$$F_r = -\frac{h(\varepsilon)E(\varepsilon)}{1-\sigma^2}\Big(\frac{\sigma}{\varepsilon R}\frac{\partial s^\varepsilon}{\partial z} + \frac{\eta^\varepsilon}{\varepsilon^2 R^2}\Big) + h(\varepsilon)\,G(\varepsilon)k(\varepsilon)\frac{\partial^2 \eta^\varepsilon}{\partial z^2} - \rho_\omega h(\varepsilon)\frac{\partial^2 \eta^\varepsilon}{\partial t^2}, \tag{2.2}$$

$$F_z = \frac{h(\varepsilon)E(\varepsilon)}{1-\sigma^2}\Big(\frac{\partial^2 s^\varepsilon}{\partial z^2} + \frac{\sigma}{\varepsilon R}\frac{\partial \eta^\varepsilon}{\partial z}\Big) - \rho_\omega h(\varepsilon)\frac{\partial^2 s^\varepsilon}{\partial t^2}, \tag{2.3}$$

Parameters	Values	Parameters	Values
$\varepsilon = \frac{r}{L}$	0.04	Wall density:ρ_ω	$1.1kg/m^2$
Characteristic radius:εR	0.004 m	Blood density:ρ	$1050kg/m^3$
Dynamics viscosity:μ	$3.4 \times 10^{-3} m^2/s$	Reference pressure:P_0	13000 Pa
Young's modulus:E	6000 Pa	Normalized pressure drop:	$\varepsilon^{1/2}$
Shear modulus:G*k	5×10^5Pa	Wall thickness:h	$4{\times}10^{-4}$m

Table 2.1: the values

Where η^ε is the radial displacement, s^ε is the longitudinal displacement, $h = h(\varepsilon)$ is the membrane thickness, ρ_ω is the wall volumetric mass, $E = E(\varepsilon)$ is the Young's modulus, $0 < \sigma < \frac{1}{2}$ is the Poisson ration, $G = G(\varepsilon)$ is the shear modulus, $k = k(\varepsilon)$ is the Timoshenko shear correction factor, F_r is the radial component of the external forces, F_z is the longitudinal component of the external forces, coming from the stresses induced by the fluid.

For the underlying blood-flow problem, the parameter values are presented in Table 2.1. Throughout the paper we will assume the following relationships between the parameters in the model.

Assumption 1 *The Young's modulus $E(\varepsilon)$, the wall thickness $h(\varepsilon)$, and the shear modulus $G(\varepsilon)k(\varepsilon)$ satisfy*

$$h(\varepsilon)E(\varepsilon) > \varepsilon, \tag{2.4}$$

$$\lim_{\varepsilon\to 0} \frac{h(\varepsilon)E(\varepsilon)}{\varepsilon} = E_0 \in (0, +\infty), \tag{2.5}$$

$$\lim_{\varepsilon\to 0} \varepsilon h(\varepsilon)G(\varepsilon)k(\varepsilon) = G_0 \in [0, +\infty). \tag{2.6}$$

Initially, the cylinder is filled with fluid and the entire structure is in an equilibrium. The equilibrium state has an initial reference pressure P_0 and the initial velocity is zero, $i.e, v^\varepsilon = 0, t = 0$. Denote the (membrane) stress tensor by T, then in the equilibrium (unperturbed) state only the T_{zz} and $T_{\vartheta\vartheta}$ components of the stress tensor T corresponding to the curved membrane Σ_ε are not zero. Their values are kG and $\varepsilon R\Delta P_0/h$ respectively, where ΔP_0 is the difference between the reference pressure in the tube and the surrounding tissue. For simplicity we assume that $\Delta P_0 = 0$, hence $T_{\vartheta\vartheta} = 0$ in the unperturbed state. The pressure difference between the inlet and the outlet boundary of Ω_ε creates a deviation from the unperturbed state. We assume that the pressure drop is small compared to the reference pressure and that the fluid acceleration is negligible compared to the effects of the fluid viscosity μ. Therefore, we can use the axially symmetric incompressible Stokes system to model fluid velocity $v^\varepsilon = (v_r^\varepsilon, v_\theta^\varepsilon, v_z^\varepsilon)$ and the pressure perturbation p^ε from the reference pressure P_0. Assuming zero angular velocity, in cylindrical coordinates the Eulerian formulation of the problem reads

$$-\mu\Big(\frac{\partial^2 v_r^\varepsilon}{\partial r^2}+\frac{\partial^2 v_r^\varepsilon}{\partial z^2}+\frac{1}{r}\frac{\partial v_r^\varepsilon}{\partial r}-\frac{v_r^\varepsilon}{r^2}\Big)+\frac{\partial p^\varepsilon}{\partial r}=0 \quad in \quad \Omega_\varepsilon\times\mathbb{R}_+ \tag{2.7}$$

$$-\mu\Big(\frac{\partial^2 v_z^\varepsilon}{\partial r^2}+\frac{\partial^2 v_z^\varepsilon}{\partial z^2}+\frac{1}{r}\frac{\partial v_z^\varepsilon}{\partial r}\Big)+\frac{\partial p^\varepsilon}{\partial z}=0 \quad in \quad \Omega_\varepsilon\times\mathbb{R}_+ \tag{2.8}$$

$$\frac{\partial v_r^\varepsilon}{\partial r}+\frac{\partial v_z^\varepsilon}{\partial z}+\frac{v_r^\varepsilon}{r}=0 \quad in \quad \Omega_\varepsilon\times\mathbb{R}_+ \tag{2.9}$$

These equations are coupled with the Navier equations for the curved membrane through the lateral boundary conditions requiring continuity of velocity and continuity of forces at the wall Σ_ε. More specifically, we require

$$v_r^\varepsilon=\frac{\partial \eta^\varepsilon}{\partial t} \quad on \quad \Sigma_\varepsilon\times\mathbb{R}_+ \tag{2.10}$$

$$v_z^\varepsilon=\frac{\partial s^\varepsilon}{\partial t} \quad on \quad \Sigma_\varepsilon\times\mathbb{R}_+ \tag{2.11}$$

and we set the radial and longitudinal forces F_r and F_z in (2.2) and (2.3) equal to the radial and longitudinal component of the stress exerted by the fluid to the membrane

$$-F_r=(p^\varepsilon I-2\mu D(v^\varepsilon))\vec{e}_r\cdot\vec{e}_r=p^\varepsilon-2\mu\frac{\partial v_r^\varepsilon}{\partial r} \quad on \quad \Sigma_\varepsilon\times\mathbb{R}_+ \tag{2.12}$$

$$-F_z=(p^\varepsilon I-2\mu D(v^\varepsilon))\vec{e}_r\cdot\vec{e}_z=-\mu\Big(\frac{\partial v_r^\varepsilon}{\partial z}+\frac{\partial v_z^\varepsilon}{\partial r}\Big) \quad on \quad \Sigma_\varepsilon\times\mathbb{R}_+ \tag{2.13}$$

where $D(v^\varepsilon)$ is the rate of the strain tensor, *i.e.*, the symmetrized gradient of the velocity

$$D(v^\varepsilon)=\frac{1}{2}(\nabla v^\varepsilon+(\nabla v^\varepsilon)^T). \tag{2.14}$$

Note: $v^\varepsilon=(v_r^\varepsilon,v_\theta^\varepsilon,v_z^\varepsilon)$.

We note that in this approximation the interface is identified with the reference elastic wall Σ_ε. The initial state of the structure is unperturbed and the initial velocity is zero

$$\eta^\varepsilon=s^\varepsilon=\frac{\partial \eta^\varepsilon}{\partial t}=\frac{\partial s^\varepsilon}{\partial t}=0 \quad on \quad \Sigma_\varepsilon\times\{0\}. \tag{2.15}$$

and we consider the following boundary data, which give rise to a well-posed initial-boundary-value problem for the limiting configuration ($\varepsilon\to 0$):

$$v_r^\varepsilon=0 \quad and \quad p^\varepsilon=0 \quad on \quad (\partial\Omega_\varepsilon\cap\{z=0\})\times\mathbb{R}_+, \tag{2.16}$$

$$v_r^\varepsilon=0 \quad and \quad p^\varepsilon=A(t) \quad on \quad (\partial\Omega_\varepsilon\cap\{z=L\})\times\mathbb{R}_+, \tag{2.17}$$

$$\frac{\partial s^\varepsilon}{\partial z}=\eta^\varepsilon=0 \; for \; z=0, \quad s^\varepsilon=\eta^\varepsilon=0 \; for \; z=L \quad and \quad \forall t\in\mathbb{R}_+. \tag{2.18}$$

Notice that the pressure drop $A(t)$ drives the problem. For simplicity we assume that $A(t)$ is smooth, i.e. that $A(t) \in C_0^\infty(0,+\infty)$. Note that physically one should expect nonzero displacements at the outlet boundary. The fixed outlet boundary, required in (2.18) gives rise to the formation of a boundary layer. Periodic boundary conditions, although natural in rigid-wall geometries, do not give rise to well-posed limiting problems when compliant walls are considered.

We summarize the initial-boundary-value problem for the coupled fluid-structure interaction driven by the time-dependent pressure drop between the inlet and the outlet boundary.

Problem 1 **(P^ε)** *For each fixed $\varepsilon > 0$, find a solution to (2.7), (2.8) and (2.9) in domain Ω_ε defined by (2.1), with an elastic lateral boundary Σ_ε. The lateral boundary conditions are given by the continuity of the velocity defined by (2.10) and (2.11) and by the continuity of the forces defined by (2.2) and (2.3), where F_r and F_z are defined in (2.12) and (2.13) respectively. The boundary conditions at the inlet and outlet boundary are defined in (2.16) and (2.17), and the behavior of the elastic wall there is prescribed by (2.18), the initial data is given by (2.15).*

2.2 Weak formulation

We define the test function space V^ε and the solution space $\mathcal{V}^\varepsilon$ as follows.

Definition 1 *The space $V^\varepsilon \subset H^1(\Omega_\varepsilon)^3$ consists of all axially symmetric functions φ such that $\varphi_r|_{\Sigma_\varepsilon}, \varphi_z|_{\Sigma_\varepsilon} \in H^1(0,L)$, $\varphi_r(0,r) = \varphi_r(L,r) = 0, \varphi_z(L,\varepsilon R) = 0$ for $r \leq \varepsilon R$ and $div\varphi = 0$ in Ω_ε.*

Definition 2 *The space $\mathcal{V}^\varepsilon$ consists of all functions $(w_r, w_z, d_r, d_z) \in H^1((0,T);V^\varepsilon) \times (H^1((0,L)\times(0,T))^2 \bigcap H^2((0,T);L^2(0,L))^2)$ such that*

1. $\frac{\partial w_r}{\partial r} + \frac{\partial w_z}{\partial z} + \frac{w_r}{r} = 0$ *in* $\Omega_\varepsilon \times \mathbb{R}_+$
2. $r^{-1} w_r \in L^2((0,T)\times\Omega_\varepsilon)$
3. $d_r(t,0) = d_z(t,L) = d_r(t,L) = 0$ *on* $\mathbb{R}_+$
4. $w_r = 0$ *on* $(\partial\Omega_\varepsilon \bigcap \{z=0\}) \times \mathbb{R}_+$
5. $w_r = 0$ *on* $(\partial\Omega_\varepsilon \bigcap \{z=L\}) \times \mathbb{R}_+$
6. $w_r = \frac{\partial d_r}{\partial t}$ *and* $w_z = \frac{\partial d_z}{\partial t}$ *on* $\Sigma_\varepsilon \times \mathbb{R}_+$.

Recall that for an axially symmetric vector valued function $\psi = \psi_r \vec{e}_r + \psi_z \vec{e}_z$, we have

$$D(\psi) = \begin{pmatrix} \frac{\partial \psi_r}{\partial r} & 0 & \frac{1}{2}\left(\frac{\partial \psi_r}{\partial z} + \frac{\partial \psi_z}{\partial r}\right) \\ 0 & \frac{\psi_r}{r} & 0 \\ \frac{1}{2}\left(\frac{\partial \psi_r}{\partial z} + \frac{\partial \psi_z}{\partial r}\right) & 0 & \frac{\partial \psi_z}{\partial z} \end{pmatrix}$$

Define the matrix norm $|\cdot|$ through the scalar product

$$\Phi : \Psi \equiv trace(\Phi \cdot \Psi^T), \Phi, \Psi \in \mathbb{R}^9.$$

Then, for each fixed $\varepsilon > 0$, the variational formulation and weak solution are defined as follows.

Definition 3 *The vector function $(v_r^\varepsilon, v_z^\varepsilon, \eta^\varepsilon, s^\varepsilon) \in \mathcal{V}^\varepsilon$ is called a weak solution of **Problem 1 (P^ε)** if the following variational formulation is satisfied:*

$$\begin{aligned} & 2\mu \int_{\Omega_\varepsilon} D(v^\varepsilon) : D(\varphi) r dr dz \\ & + \varepsilon R \int_0^L \left\{ h(\varepsilon) G(\varepsilon) k(\varepsilon) \frac{\partial \eta^\varepsilon}{\partial z} \frac{\partial \varphi_r}{\partial z} + \frac{h(\varepsilon) E(\varepsilon)}{1-\sigma^2} \big(\frac{\sigma}{\varepsilon R} \frac{\partial s^\varepsilon}{\partial z} + \frac{\eta^\varepsilon}{\varepsilon^2 R^2} \big) \varphi_r \right. \\ & \left. + \frac{h(\varepsilon) E(\varepsilon)}{1-\sigma^2} \big(\frac{\partial s^\varepsilon}{\partial z} \frac{\partial \varphi_z}{\partial z} - \frac{\sigma}{\varepsilon R} \frac{\partial \eta^\varepsilon}{\partial z} \varphi_z \big) \right\} \Big|_{r=\varepsilon R} dz + \varepsilon R \rho_\omega h(\varepsilon) \frac{d^2}{dt^2} \int_0^L (\eta^\varepsilon \varphi_r + s^\varepsilon \varphi_z)|_{r=\varepsilon R} dz \\ & = - \int_0^{\varepsilon R} A(t) \varphi_z|_{z=L} r dr \quad \forall \varphi = \varphi_r \vec{e}_r + \varphi_z \vec{e}_z \in V^\varepsilon \end{aligned} \tag{2.19}$$

with

$$v_r^\varepsilon = \frac{\partial \eta^\varepsilon}{\partial t} \quad on \quad \Sigma_\varepsilon \times \mathbb{R}_+ \tag{2.20}$$

$$v_z^\varepsilon = \frac{\partial s^\varepsilon}{\partial t} \quad on \quad \Sigma_\varepsilon \times \mathbb{R}_+ \tag{2.21}$$

and initial conditions

$$\eta^\varepsilon = s^\varepsilon = \frac{\partial \eta^\varepsilon}{\partial t} = \frac{\partial s^\varepsilon}{\partial t} = 0 \quad on \quad \Sigma_\varepsilon \times \{0\}. \tag{2.22}$$

After attaining the variational formulation, we can state the existence theorem for our fluid-structure problem.

Theorem 1 *For every $\varepsilon > 0$, there exists a unique solution $(v_r^\varepsilon, v_z^\varepsilon, \eta^\varepsilon, s^\varepsilon) \in \mathcal{V}^\varepsilon$ to (2.19).*

2.3 Existence and Uniqueness of P^ε

To show the existence and uniqueness of the solution to the coupled problem, we will employ the widely-used *Galerkin approach.*

There are three cases: see [14]

- A *fixed* interface. There is no displacement or the displacements are infinitesimal, so that the interface condition simply reduced to *no-slip* condition.
- A *moving* interface. There exits large stress-induced displacements. And this is not known generally and must be found as part of the solution process.
- The interface is in a situation between *fixed* and *moving.* This is the case we will study here.

First we simplified our domain, which is easy to deal with. Let Ω_1 be the fluid domain, Ω_2 be the structure domain, Γ_0 be the interface between Ω_1 and Ω_2, i.e. $\Gamma_0 = \partial\Omega_1 \cap \partial\Omega_2$.

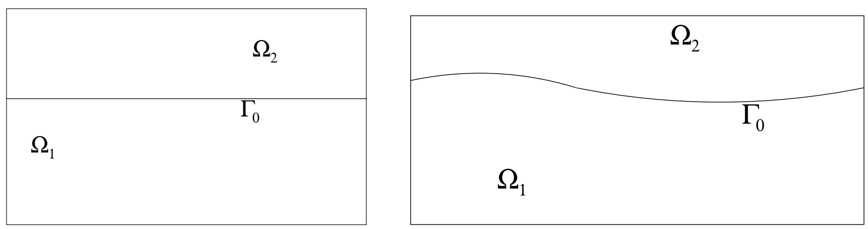


Figure 2.2: The domain for the fluid and vessel, $\Gamma_0 = \partial\Omega_1 \cap \partial\Omega_2$

Recall the coupled equations

$$\begin{cases} -\mu\left(\frac{\partial^2 v_r^\varepsilon}{\partial r^2} + \frac{\partial^2 v_r^\varepsilon}{\partial z^2} + \frac{1}{r}\frac{\partial v_r^\varepsilon}{\partial r} - \frac{v_r^\varepsilon}{r^2}\right) + \frac{\partial p^\varepsilon}{\partial r} = 0 \quad in \quad \Omega_\varepsilon \times \mathbb{R}_+ \\ -\mu\left(\frac{\partial^2 v_z^\varepsilon}{\partial r^2} + \frac{\partial^2 v_z^\varepsilon}{\partial z^2} + \frac{1}{r}\frac{\partial v_z^\varepsilon}{\partial r}\right) + \frac{\partial p^\varepsilon}{\partial z} = 0 \quad in \quad \Omega_\varepsilon \times \mathbb{R}_+ \\ \frac{\partial v_r^\varepsilon}{\partial r} + \frac{\partial v_z^\varepsilon}{\partial z} + \frac{v_r^\varepsilon}{r} = 0 \quad in \quad \Omega_\varepsilon \times \mathbb{R}_+ \\ v^\varepsilon(r,z,0) = 0 \end{cases} \tag{2.23}$$

$$\begin{cases} F_r = -\frac{h(\varepsilon)E(\varepsilon)}{1-\sigma^2}\left(\frac{\sigma}{\varepsilon R}\frac{\partial s^\varepsilon}{\partial z} + \frac{\eta^\varepsilon}{\varepsilon^2 R^2}\right) + h(\varepsilon)G(\varepsilon)k(\varepsilon)\frac{\partial^2\eta^\varepsilon}{\partial z^2} - \rho_\omega h(\varepsilon)\frac{\partial^2\eta^\varepsilon}{\partial t^2}, \\ F_z = \frac{h(\varepsilon)E(\varepsilon)}{1-\sigma^2}\left(\frac{\partial^2 s^\varepsilon}{\partial z^2} + \frac{\sigma}{\varepsilon R}\frac{\partial\eta^\varepsilon}{\partial z}\right) - \rho_\omega h(\varepsilon)\frac{\partial^2 s^\varepsilon}{\partial t^2}, \\ s^\varepsilon = \eta^\varepsilon = 0, \quad t = 0 \end{cases} \tag{2.24}$$

$continuity\ of\ force,$

$$\begin{cases} -F_r = p^\varepsilon - 2\mu\frac{\partial v_r^\varepsilon}{\partial r} \quad on \quad \Sigma_\varepsilon \times \mathbb{R}_+ \\ -F_z = -\mu\left(\frac{\partial v_r^\varepsilon}{\partial z} + \frac{\partial v_z^\varepsilon}{\partial r}\right) \quad on \quad \Sigma_\varepsilon \times \mathbb{R}_+ \end{cases} \tag{2.25}$$

$continuity\ of\ velocity,$

$$\begin{cases} v_r^\varepsilon = \frac{\partial\eta^\varepsilon}{\partial t} \quad on \quad \Sigma_\varepsilon \times \mathbb{R}_+ \\ v_z^\varepsilon = \frac{\partial s^\varepsilon}{\partial t} \quad on \quad \Sigma_\varepsilon \times \mathbb{R}_+ \end{cases} \tag{2.26}$$

First we will introduce the whole coupled equations as a system. Let

$$\begin{array}{lll} u_{r1} = \int_0^t v_r^\varepsilon dt, & u_{r2} = \dfrac{\partial \eta^\varepsilon}{\partial t} & \Rightarrow u_r = (u_{r1}, u_{r2}) \\ u_{z1} = \int_0^t v_z^\varepsilon dt, & u_{z2} = \dfrac{\partial s^\varepsilon}{\partial t} & \Rightarrow u_z = (u_{z1}, u_{z2}) \\ \quad\Downarrow & \quad\Downarrow & \\ u_1 = (u_{r1}, u_{z1}), & u_2 = (u_{r2}, u_{z2}), & \Rightarrow u = (u_1, u_2) \end{array}$$

In the future, we will multiply the coupled equation with φ, which will also have the same definition corresponding to the above.

So we have :

$$\begin{array}{l} v_r^\varepsilon = u'_{r1}, \quad v_z^\varepsilon = u'_{z1} \\ \dfrac{\partial^2 \eta^\varepsilon}{\partial t^2} = u'_{r2}, \quad \eta^\varepsilon = \displaystyle\int_0^t u_{r2} dt \\ \dfrac{\partial^2 s^\varepsilon}{\partial t^2} = u'_{z2}, \quad s^\varepsilon = \displaystyle\int_0^t u_{z2} dt \\ continuity\ of\ the\ velocity\ : u'_{r1} = u_{r2}, \quad u'_{z1} = u_{z2} \end{array}$$

Note: $u' = \frac{\partial u}{\partial t}$. And we will use the same notation in the following sections.

The system becomes :

$$\begin{cases} -\mu\Big(\frac{\partial^2 u'_{r1}}{\partial r^2} + \frac{\partial^2 u'_{r1}}{\partial z^2} + \frac{1}{r}\frac{\partial u'_{r1}}{\partial r} - \frac{u'_{r1}}{r^2}\Big) + \frac{\partial p^\varepsilon}{\partial r} = 0 \quad in \quad \Omega_\varepsilon \times \mathbb{R}_+ \\ -\mu\Big(\frac{\partial^2 u'_{z1}}{\partial r^2} + \frac{\partial^2 u'_{z1}}{\partial z^2} + \frac{1}{r}\frac{\partial u'_{z1}}{\partial r}\Big) + \frac{\partial p^\varepsilon}{\partial z} = 0 \quad in \quad \Omega_\varepsilon \times \mathbb{R}_+ \\ \nabla \cdot u'_1 = 0 \quad in \quad \Omega_\varepsilon \times \mathbb{R}_+ \\ u'_1 = 0, \ t = 0 \end{cases} \tag{2.27}$$

$$\begin{cases} F_r = -\frac{h(\varepsilon)E(\varepsilon)}{1-\sigma^2}\Big(\frac{\sigma}{\varepsilon R}\frac{\partial}{\partial z}(\int_0^t u_{z2}dt) + \frac{1}{\varepsilon^2 R^2}(\int_0^t u_{r2}dt)\Big) + h(\varepsilon)G(\varepsilon)k(\varepsilon)\frac{\partial^2}{\partial z^2}(\int_0^t u_{r2}dt) - \rho_\omega h(\varepsilon)u'_{r2}, \\ F_z = \frac{h(\varepsilon)E(\varepsilon)}{1-\sigma^2}\Big(\frac{\partial^2}{\partial z^2}(\int_0^t u_{z2}dt) + \frac{\sigma}{\varepsilon R}\frac{\partial}{\partial z}(\int_0^t u_{r2}dt)\Big) - \rho_\omega h(\varepsilon)u'_{z2}, \\ u_{r2} = u_{z2} = 0, \quad t = 0 \end{cases} \tag{2.28}$$

$continuity\ of\ force,$

$$\begin{cases} -F_r = p^\varepsilon - 2\mu\frac{\partial u'_{r1}}{\partial r} \quad on \quad \Sigma_\varepsilon \times \mathbb{R}_+ \\ -F_z = -\mu(\frac{\partial u'_{r1}}{\partial z} + \frac{\partial u'_{z1}}{\partial r}) \quad on \quad \Sigma_\varepsilon \times \mathbb{R}_+ \end{cases} \tag{2.29}$$

$continuity\ of\ velocity,$

$$\begin{cases} u'_{r1} = u_{r2} \quad on \quad \Sigma_\varepsilon \times \mathbb{R}_+ \\ u'_{z1} = u_{z2} \quad on \quad \Sigma_\varepsilon \times \mathbb{R}_+ \end{cases} \tag{2.30}$$

Variational formulation :

Bilinear forms

- $a_1(u_1,\varphi_1) = 2\mu \int_{\Omega_\varepsilon} D(u_1) : D(\varphi_1) d\Omega_\varepsilon$
-

$$\begin{aligned} a_2(u_2,\varphi_2) &= \varepsilon R \int_0^L \Big\{ h(\varepsilon)G(\varepsilon)k(\varepsilon)\frac{\partial}{\partial z}(\int_0^t u_{r2}dt)\frac{\partial \varphi_{r2}}{\partial z} \\ &+\frac{h(\varepsilon)E(\varepsilon)}{1-\sigma^2}\Big(\frac{\sigma}{\varepsilon R}\frac{\partial}{\partial z}(\int_0^t u_{z2}dt) + \frac{1}{\varepsilon^2 R^2}(\int_0^t u_{r2}dt)\Big)\varphi_{r2} \\ &+\frac{h(\varepsilon)E(\varepsilon)}{1-\sigma^2}\Big(\frac{\partial}{\partial z}(\int_0^t u_{z2}dt)\frac{\partial \varphi_{z2}}{\partial z} \underbrace{-\frac{\sigma}{\varepsilon R}\frac{\partial}{\partial z}(\int_0^t u_{r2}dt)\varphi_{z2}\Big)}_{\heartsuit}\Big\}dz \end{aligned}$$

- $(u_2', \varphi_2) = \varepsilon R \rho_\omega h(\varepsilon) \int_0^L \Big(u_{r2}'\varphi_{r2} + u_{z2}'\varphi_{z2}\Big)dz$
- $(A(t), \varphi_{z1}) = \int_0^{\varepsilon R} A(t)\varphi_{z1}|_{z=L} r dr,$
- $a(u,\varphi) = a_1(u_1,\varphi_1) + a_2(u_2,\varphi_2)$

Find $u \in V^\varepsilon$ such that :

$$\begin{cases} (u_2',\varphi_2) + a(u,\varphi) + (A(t),\varphi_{z1}) = 0, \ \forall \varphi \in \mathcal{V}^\varepsilon \\ u = 0, \quad t = 0 \end{cases} \tag{2.31}$$

boundary conditions

$$\begin{cases} p^\varepsilon = 0 \ and \ u_{r1}' = 0, \ on \ (\partial\Omega_\varepsilon \cap \{z=0\}) \times \mathbb{R}_+ \\ p^\varepsilon = A(t) \ and \ u_{r1}' = 0, \ on \ (\partial\Omega_\varepsilon \cap \{z=L\}) \times \mathbb{R}_+ \\ \frac{\partial}{\partial z}(\int_0^t u_{z2}dt) = \int_0^t u_{r2}dt = 0, \ for \ z = 0, \forall t \in \mathbb{R}_+ \\ \int_0^t u_{z2}dt = \int_0^t u_{r2}dt = 0, \ for \ z = L, \forall t \in \mathbb{R}_+ \end{cases} \tag{2.32}$$

The weak existence theory for (2.31) and (2.32) requires that $a_1(u_1,\varphi_1), a_2(u_2,\varphi_2)$ and $(A(t),\varphi_{z1})$ are continuous (bounded) and that $a_1(u_1,\varphi_1), a_2(u_2,\varphi_2)$ are coercive (positive definite). We have no doubt that operator a_1 is bounded and coercive, what's more, we have from next chapter, energy estimate, we can see that operator a_2 is bounded, now the problem is that whether operator a_2 is coercive or not? We found that operator a_2 is indeed positive definite for the case of small deformations (linear elastic approximation), then by the *Hydrodynamic Lemma*, the $\heartsuit$ term will be small enough or vanish.

The rest steps are similar to the next section, the existence and uniqueness of the solution to **FSI** problems.

2.4 Existence and Uniqueness of the solution to FSI problems

Here we will use the widely-used *Galerkin approach* to show the existence of a solution the weak formulation (2.19).

First we enlarge the area near the boundary, then we set fluid domain as Ω_1, the lateral wall as Ω_2, these two meet on the surface Γ_0, i.e. $\Gamma_0 = \Omega_1 \cap \Omega_2$, we denote that $\Gamma_1 = \partial\Omega_1 \backslash \Gamma_0$, $\Gamma_2 = \partial\Omega_2 \backslash \Gamma_0$

Governing equations

$$\begin{cases} \frac{\partial u}{\partial t} - \mu \Delta u = f_1 - \nabla p, \ in \ \ Q_1 = \Omega_1 \times (0,T) \\ \nabla \cdot u = 0, \ in \ \ Q_1 \\ u(x,0) = u_0(x), \ in \ \ \Omega_1 \end{cases} \tag{2.33}$$

$$\begin{cases} \frac{\partial^2 s}{\partial^2 t} - \Delta s = f_2, \ in \ \ Q_2 = \Omega_2 \times (0,T) \\ s(x,0) = s_0, \ \frac{\partial s}{\partial t}(x,0) = s_1 \end{cases} \tag{2.34}$$

Transmission conditions, continuity of velocity and the force

$$\begin{cases} u = \frac{\partial s}{\partial t}, \ on \ \Gamma_0, \ continuity \ of \ velocity, \\ \mu \nabla u \cdot n_1 - p n_1 = \nabla s \cdot n_2, \ on \ \Gamma_0, \ continuity \ of \ force. \end{cases} \tag{2.35}$$

The coupled equations as a system : Let $u_1 = u, u_2 = \frac{\partial s}{\partial t}$ so that $s(x,t) = \int_0^t u_2(x,\sigma) d\sigma + s_0(x)$.

The system becomes :

$$\begin{cases} \frac{\partial u_1}{\partial t} - \mu \Delta u_1 = f_1 - \nabla p, \nabla \cdot u_1 = 0, \ in \ Q_1, \\ \frac{\partial u_2}{\partial t} - \Delta(\int_0^t u_2 d\sigma) = f_2 + \Delta s_0, \ in \ Q_2, \\ u_1 = u_0(x), \ \ in \ \ \Omega_1 \\ u_2 = s_1(x), \ \ in \ \ \Omega_2 \end{cases} \tag{2.36}$$

The transmission conditions read :

$$\begin{cases} u_1 = u_2, \ on \ \Gamma_0 \\ \mu \nabla u_1 \cdot n_1 - p n_1 = \int_0^t \nabla u_2 \cdot n_2 d\sigma + \nabla s_0 \cdot n_2 \end{cases} \tag{2.37}$$

Variational formulation :

We let

- Let Ω be the interior of $\overline{\Omega_1} \cup \overline{\Omega_2}$,
- $v = \{v_1, v_2\}$, $v_i = v|_{\Omega_i}, i = 1, 2$,
- $V = \{v \in (H_0^1(\Omega))^n, \nabla \cdot v = 0, \ \ in \ \ \Omega_1, \ \ \ v_1 = v_2 \ \ on \ \ \Gamma_0\}$.

Bilinear forms :

- $a_1(u_1, v_1) = \mu \int_{\Omega_1} \nabla u_1 \cdot \nabla v_1 dx$,
- $a_2(u_2, v_2) = \int_{\Omega_2} \nabla u_2 \cdot \nabla v_2 dx$
- $(f, v) = (f_1, v_1)_1 + (f_2, v_2)_2 - a_2(s_0, v_2)$,
- $(f_i, v_i)_i = \int_{\Omega_i} f_i v_i dx$,
- $(u, v) = (u_1, v_1)_1 + (u_2, v_2)_2$

Find $u = u(t) \in V$ such that :

$$\begin{cases} (u', v) + a_1(u_1, v_1) + a_2(\int_0^t u_2 d\sigma, v_2) = (f, v), \quad \forall v \in V, \\ u(0) = \{u_0, s_1\} \end{cases} \tag{2.38}$$

The weak existence theory for (2.38) requires that $a_1(u, v), a_2(u, v), (f, v)$ are continuous (bounded) and that $a_1(u, v), a_2(u, v)$ are coercive (positive definite).

We have :

$$|a_1(u_1, v_1)| \leq C_1 \|u_1\|_V \|v_1\|_V$$
$$|a_2(u_2, v_2)| \leq C_2 \|u_2\|_V \|v_2\|_V$$

By the *Korn inequality* we also have

$$a_1(u_1, u_1) \geq C_3 \|u_1\|_V^2$$
$$a_2(u_2, u_2) \geq C_4 \|u_1\|_V^2$$

We now consider the equation (2.38) as a parabolic problem,

$$\begin{cases} u' + \mathbf{A}u = f, \quad in \quad Q = \Omega \times (0, T), \\ u = u_0, \quad on \quad \Omega \times \{t = 0\} \\ u = 0, \quad on \quad \partial\Omega \times (0, T) \end{cases} \tag{2.39}$$

Next step, we will show the existence of the weak solution to this problem, but we will not give it in details, only give a sketch of it. Mainly here we will employ the widely-used *Galerkin approximation.* Let $\{w_k\}_{k=1}^{\infty}$ be orthonormal basis with respect to the inner product in V.

Fix m and define

$$\begin{cases} u_m(t) = \sum_{k=1}^{m} \xi_k(t) w_k \\ \xi_k(0) = (u_0, w_k), k = 1, \cdots, m \end{cases} \tag{2.40}$$

from this we get :

$$\begin{cases} a_1(u_m(t), w_k) = \sum_{l=1}^{m} e_{kl}^1 \xi_k(t) \\ a_2(u_m(t), w_k) = \sum_{l=1}^{m} e_{kl}^2 \int_0^t \xi_k(\sigma) d\sigma \\ where \quad e_{kl}^1 = a_1(w_k, w_l), \quad e_{kl}^2 = a_2(w_k, w_l) \end{cases} \tag{2.41}$$

and we define a new operator a as the following, $\phi_k(t) = \int_0^t \xi_k(\sigma)d\sigma$,

$$a(u_m(t), w_k) = a_1(u_m(t), w_k) + a_2(u_m(t), w_k)$$

where u_m is the projection of u onto the m dimensional subspace V_m of V spanned by $\{w_k\}_{k=1}^m$. Since $(u_m(t), w_k) = \xi_k(t)$, termwise differentiation yields $(u'_m(t), w_k) = \xi'_k(t)$, we also put $f_k(t) = (f(t), w_k), k = 1, \cdots, m$.

We obtain a linear system of ODE's in $(0, T)$

$$\begin{cases} \xi'_k(t) + \sum_{l=1}^m e_{kl}^1 \xi_k(t) + \sum_{l=1}^m e_{kl}^2 \phi_k(t) = f_k(t), k = 1, \cdots, m. \\ \phi'_k(t) = \xi_k(t), \\ \xi_k(0) = (u_0(x), w_k), \\ \phi_k(0) = 0 \end{cases} \tag{2.42}$$

The matrices $M_1 = (e_{kl}^1)_{mm}$ and $M_2 = (e_{kl}^2)_{mm}$ are symmetric and positive definite and standard theory for ODE's yields the existence of a unique vector $\xi(t) = \begin{pmatrix} \xi_1(t) \\ \vdots \\ \xi_m(t) \end{pmatrix}$ and thus unique $u_m(t) = \sum_{k=1}^m \xi_k(t) w_k$ we then study the variational problem for fixed m

$$(u'_m, w_k) + a(u_m, w_k) = (f, w_k) \tag{2.43}$$

2.4.1 Uniform estimates–a priori estimates

We have that

$$max_{0 \le t \le T} \|u_m(t)\|_{L^2(\Omega)} + \|u_m\|_{L^2(V \times (0,T))} + \|u'_m\|_{L^2(V^1 \times (0,T))} \le C(\|f\| + \|u_0\|)$$

here the norms of forcing and intial data are $\|f\|_{L^2(\Omega \times (0,T))}$ and $\|u_0\|_{L^2(\Omega)}$.

Proof :

Take the Galerkin equation :

$$(u'_m, w_k) + a(u_m, w_k) = (f, w_k)$$

multiply by $\xi_k(t)$ to get

$$(u'_m, u_m) + a(u_m, u_m) = (f, u_m)$$

by the coercivity

$$a(u_m, u_m) \ge C\|u_m\|_V^2, C > 0$$
$$|(f, u_m)| \le \frac{1}{2}\|f\|_{L^2(\Omega)}^2 + \frac{1}{2}\|u_m(t)\|_{L^2(\Omega)}^2.$$

we get

$$\frac{d}{dt}\|u_m(t)\|^2_{L^2(\Omega)} + 2C\|u_m(t)\|^2_V \le C_1\|u_m(t)\|^2_{L^2(\Omega)} + C_2\|f\|^2_{L^2(\Omega)},$$

$$\Rightarrow \begin{cases} max_{0\le t\le T}\|u_m(t)\|^2_{L^2(\Omega)} \le C\Big(\|u_0\|^2 + \|f\|^2\Big), & \text{Gronwall Inequality} \\ \|u_m\|^2_{L^2(V\times(0,T))} \le C\Big(\|u_0\|^2 + \|f\|^2\Big), & \text{Poincaré Inequality} \end{cases}$$

Estimate for time derivative : Fix any $v \in H^1_0(\Omega)$ with $\|v\|_{H^1_0(\Omega)} \le 1$ and write $v = v_1 + v_2$ with $v_1 \in Span\{w_k\}_{k=1}^m$, $(v_2, w_k) = 0, k = 1, \cdots, m$, obviously

$$\|v_1\|_{H^1_0(\Omega)} \le \|v\|_{H^1_0(\Omega)} \le 1$$

we get

$$\langle u'_m, v\rangle = (u'_m, v) = (u'_m, v_1) = (f, v_1) - a(u_m, v_1)$$

using continuity of $a(\cdot,\cdot) and \|v_1\| \le 1$, and we can obtain

$$\begin{aligned} &|\langle u'_m, v\rangle| \le C(\|f\|_{L^2(\Omega)} + \|u_m\|_{H^1_0(\Omega)}), \\ &\Rightarrow \|u'_m\|_{H^{-1}(\Omega)} \le C(\|f\|_{L^2(\Omega)} + \|u_m\|_{H^1_0(\Omega)}) \\ &\Rightarrow \int_0^T \|u'_m\|_{H^{-1}(\Omega)} dt \le C(\|f\|^2_{L^2(\Omega)} + \|u_0\|^2_{L^2(\Omega)}) \end{aligned}$$

By the a priori estimates

$$\begin{aligned} &\|u_m\|_{L^2(V\times(0,T))} \le C, \\ &\|u'_m\|_{L^2(V^1\times(0,T))} \le C, \\ &\|u_m\|_{L^\infty(L^2(\Omega)\times(0,T))} \le C, \end{aligned}$$

we conclude that there exists a (sub)sequence $\{u_{m_k}\}_{k=1}^\infty \subset \{u_m\}_{m=1}^\infty$ and a function $u \in V$:

$$\begin{aligned} &u_{m_k} \rightharpoonup u, \quad in \quad V, \\ &u'_{m_k} \rightharpoonup u', \quad in \quad V^1 \end{aligned}$$

Identification of limit : Fix N and let

$$v(t) = \sum_{k=1}^N v_k(t) w_k \tag{2.44}$$

$\{v_k(t)\}_{k=1}^N$ is smooth in $[0, T)$. Multiply the Galerkin equation by v as in (2.44) and integrate from 0 to T,

$$\int_0^T \Big(\langle u'_m(\sigma), v(\sigma)\rangle + a(u_m(\sigma), v(\sigma))\Big) d\sigma = \int_0^T \langle f(\sigma), v(\sigma)\rangle d\sigma \tag{2.45}$$

let $m = m_k$ and pass to the limit:

$$\int_0^T \Big(\langle u'(\sigma), v(\sigma)\rangle + a(u(\sigma), v(\sigma))\Big) d\sigma = \int_0^T \langle f(\sigma), v(\sigma)\rangle d\sigma \tag{2.46}$$

recall that $u(0) = u_0$ and from (2.46) we have

$$\int_0^T (-\langle v', u\rangle + a(u, v))dt = \int_0^T (f, v)dt + (u(0), v(0)) \tag{2.47}$$

for all v as in (2.44) with $v(T) = 0$. From (2.45) we have

$$\int_0^T (-\langle v', u_m\rangle + a(u_m, v))dt = \int_0^T (f, v)dt + (u(0), v(0)) \tag{2.48}$$

a limit passage in (2.48), using $u_m(0) \rightharpoonup u_0$ in $L^2(\Omega)$ gives the result.

Uniqueness

Enough to show that for $f = u_0 \equiv 0$, trivial solution, $u \equiv 0$ is unique. Put $v = u$ in (2.46)

$$\|u\|^2_{L^\infty(L^2(\Omega)\times(0,T))} + \|u\|^2_{L^2(V\times(0,T))} \leq 0 \tag{2.49}$$

shows that $u = 0$ is a solution. $u \neq 0$ contradicts (2.49).

Chapter 3

Energy estimates

The energy of this problem, obtained by using the velocity field as a test function in (2.19) consists of the elastic energy of the membrane, the viscous energy of the fluid, and the energy due to the external forces.

We define the time derivative of the elastic energy as :

$$\begin{aligned}\frac{d\mathcal{E}_{elastic}}{dt} \equiv\ & \varepsilon R \int_0^L \Big\{ h(\varepsilon)G(\varepsilon)k(\varepsilon)\frac{\partial \eta^\varepsilon}{\partial z}\frac{\partial^2 \eta^\varepsilon}{\partial z \partial t} + \frac{h(\varepsilon)E(\varepsilon)}{1-\sigma^2}\Big(\Big(\frac{\sigma}{\varepsilon R} + \frac{\eta^\varepsilon}{\eta^2 R^2}\Big)\frac{\partial \eta^\varepsilon}{\partial t} \\ & + \Big(\frac{\partial s^\varepsilon}{\partial z}\frac{\partial^2 s^\varepsilon}{\partial z \partial t} - \frac{\sigma}{\varepsilon R}\frac{\partial \eta^\varepsilon}{\partial z}\frac{\partial s^\varepsilon}{\partial t}\Big)\Big) + \rho_\omega h(\varepsilon)\Big(\frac{\partial^2 \eta^\varepsilon}{\partial t^2}\frac{\partial \eta^\varepsilon}{\partial t} + \frac{\partial^2 s^\varepsilon}{\partial t^2}\frac{\partial s^\varepsilon}{\partial t}\Big)\Big\} dz \qquad (3.1)\end{aligned}$$

which also can be expressed as follows.

Lemma 1 *The displacements η^ε and s^ε satisfy*

$$\begin{aligned}\frac{d\mathcal{E}_{elastic}}{dt} \equiv\ & \varepsilon R \frac{d}{2dt}\Big\{ \rho_\omega h(\varepsilon) \int_0^L \Big(\Big|\frac{\partial \eta^\varepsilon}{\partial t}\Big|^2 + \Big|\frac{\partial s^\varepsilon}{\partial t}\Big|^2 \Big) dz + h(\varepsilon)G(\varepsilon)k(\varepsilon) \int_0^L \Big|\frac{\partial \eta^\varepsilon}{\partial z}\Big|^2 dz \\ & + \frac{h(\varepsilon)E(\varepsilon)}{1-\sigma^2} \int_0^L \Big[\sigma\Big(\frac{\partial s^\varepsilon}{\partial z} + \frac{\eta^\varepsilon}{\varepsilon R}\Big)^2 + (1-\sigma)\Big(\Big|\frac{\eta^\varepsilon}{\varepsilon R}\Big|^2 + \Big|\frac{\partial s^\varepsilon}{\partial z}\Big|^2 \Big) \\ & + \sigma\Big(\frac{\partial \eta^\varepsilon}{\partial z} - \frac{s^\varepsilon}{\varepsilon R}\Big)^2 - \sigma\Big(\Big|\frac{\partial \eta^\varepsilon}{\partial z}\Big|^2 + \Big|\frac{s^\varepsilon}{\varepsilon R}\Big|^2 \Big) \Big] dz \Big\} \qquad (3.2)\end{aligned}$$

We are interested in the oscillations of the membrane that are due to the time-dependent pressure drop $A(t)$. These occur at a different time-scale than the characteristic *physical* time. In particular, as we will see later that the fluid velocity is greater than the velocity of the displacement. This, in turn, gives rise to long-wavelength elastic waves. It is these waves that we would like to keep in our asymptotic reduced problem. Therefore we introduce a new time-scale

$$\tilde{t} = \omega^\varepsilon t,$$

where the characteristic frequency ω^ε will be specified later, in order to include both the waves that occur at the leading order time-scale as well as the oscillations of the membrane caused by a response of the elastic material. The pressure drop is supposed to be a function of $\tilde{t}$.

From now on we will use the rescaled time $\tilde{t}$ and drop the tilde. By keeping the rescaled time in mind and by using the expression for the elastic energy, we obtain the following.

Lemma 2 *The solution $(v_r^\varepsilon, v_z^\varepsilon, \eta^\varepsilon, s^\varepsilon)$ to (2.19) satisfies the variational equality*

$$\omega^\varepsilon h(\varepsilon)\frac{d}{2dt}\Bigg\{(\omega^\varepsilon)^2\varepsilon R\rho_\omega\Big(\|\frac{\partial\eta^\varepsilon}{\partial t}\|^2_{L_2(0,L)}+\|\frac{\partial s^\varepsilon}{\partial t}\|^2_{L_2(0,L)}\Big)+k(\varepsilon)G(\varepsilon)\varepsilon R\|\frac{\partial\eta^\varepsilon}{\partial z}\|^2_{L_2(0,L)}$$
$$+\frac{E(\varepsilon)\varepsilon R}{1-\sigma^2}\Big[\sigma\|\frac{\partial s^\varepsilon}{\partial z}+\frac{\eta^\varepsilon}{\varepsilon R}\|^2_{L_2(0,L)}+(1-\sigma)\Big(\|\frac{\eta^\varepsilon}{\varepsilon R}\|^2_{L_2(0,L)}+\|\frac{\partial s^\varepsilon}{\partial z}\|^2_{L_2(0,L)}\Big)$$
$$+\sigma\|\frac{\partial\eta^\varepsilon}{\partial z}-\frac{s^\varepsilon}{\varepsilon R}\|^2_{L_2(0,L)}-\sigma\Big(\|\frac{\partial\eta^\varepsilon}{\partial z}\|^2_{L_2(0,L)}+\|\frac{s^\varepsilon}{\varepsilon R}\|^2_{L_2(0,L)}\Big)\Big]\Bigg\}$$
$$+2\mu\int_{\Omega_\varepsilon}\big\|D(v^\varepsilon)\big\|_F^2 r\,dr\,dz=-\int_0^{\varepsilon R}A(t)v_z^\varepsilon(t,r,L)r\,dr \tag{3.3}$$

with $v_r^\varepsilon=\omega^\varepsilon\frac{\partial\eta^\varepsilon}{\partial t}$ *and* $v_z^\varepsilon=\omega^\varepsilon\frac{\partial s^\varepsilon}{\partial t}$ *on* $\Sigma_\varepsilon\times(0,T)$.

We now investigate how the energy of the forcing term controls the elastic and the viscous energy of the coupled fluid-structure interaction. We start to transfer and estimate the right-hand side. Since we do not have *no-slip* condition for the velocity at the lateral boundary, the situation is more complicated than in the derivation of *Reynolds' equation.* Furthermore, since on the LHS we only have the L^2-norm of $D(v^\varepsilon)$ and not the L^2-norm of the ∇v^ε, a standard approach based on using the *Gronwall estimate* and the L^2-norm of the velocity, $\rho\int_{\Omega_\varepsilon}|v^\varepsilon(t)|^2 r\,dr\,dz$, is not sufficient to guarantee the correct order of magnitude of the velocity. To get around this difficulty we transfer the right-hand side of (3.3) to a combination of a volume term and a lateral boundary term by integrating (2.9)$\times z$ in Ω_ε and both sides we multiply by $A(t)$. Then we get the following identity

$$-\int_0^{\varepsilon R}A(t)v_z^\varepsilon(t,r,L)r\,dr=-\int_{\Omega_\varepsilon}\frac{A(t)}{L}v_z^\varepsilon r\,dr\,dz+\varepsilon R\int_0^L A(t)\frac{z}{L}v_r^\varepsilon(t,\varepsilon R,z)dz \tag{3.4}$$

Then we get new variational equality by replacing the right-hand side by above equality.

Proposition 1 *The solution $(v_r^\varepsilon, v_z^\varepsilon, \eta^\varepsilon, s^\varepsilon)$ to (2.19) satisfies the variational equality*

$$
\begin{aligned}
&\omega^\varepsilon h(\varepsilon)\frac{d}{2dt}\Big\{(\omega^\varepsilon)^2\varepsilon R\rho_\omega\Big(\|\frac{\partial\eta^\varepsilon}{\partial t}\|^2_{L_2(0,L)} + \|\frac{\partial s^\varepsilon}{\partial t}\|^2_{L_2(0,L)}\Big) + k(\varepsilon)G(\varepsilon)\varepsilon R\|\frac{\partial \eta^\varepsilon}{\partial z}\|^2_{L_2(0,L)} \\
&+\frac{E(\varepsilon)\varepsilon R}{1-\sigma^2}\Big[\sigma\|\frac{\partial s^\varepsilon}{\partial z} + \frac{\eta^\varepsilon}{\varepsilon R}\|^2_{L_2(0,L)} + (1-\sigma)\Big(\|\frac{\eta^\varepsilon}{\varepsilon R}\|^2_{L_2(0,L)} + \|\frac{\partial s^\varepsilon}{\partial z}\|^2_{L_2(0,L)}\Big) \\
&+\sigma\|\frac{\partial\eta^\varepsilon}{\partial z} - \frac{s^\varepsilon}{\varepsilon R}\|^2_{L_2(0,L)} - \sigma\Big(\|\frac{\partial\eta^\varepsilon}{\partial z}\|^2_{L_2(0,L)} + \|\frac{s^\varepsilon}{\varepsilon R}\|^2_{L_2(0,L)}\Big)\Big]\Big\} + 2\mu\int_{\Omega_\varepsilon}\|D(v^\varepsilon)\|^2_F r\,dr\,dz \\
&= -\int_{\Omega_\varepsilon}\frac{A(t)}{L}v_z^\varepsilon r\,dr\,dz + \varepsilon R\int_0^L A(t)\frac{z}{L}v_r^\varepsilon(t,\varepsilon R,z)dz
\end{aligned}
\tag{3.5}
$$

Recall (2.10), $v_r^\varepsilon(t,\varepsilon R,z) = \frac{\partial\eta^\varepsilon}{\partial t}$.

We will use the following strategy :

- First, we add an auxiliary term

$$\omega^\varepsilon\frac{\varepsilon^2R^2}{2}A(t)\frac{\partial}{\partial t}(\int_0^L s^\varepsilon(t,z)dz) \tag{3.6}$$

  on both sides of equation (3.5), which helps to estimate the energy of axial displacement.

- Second, since this is an equality, and our goal is to estimate the energy of the whole system, the way to do this is like this: $\cdots \leqslant \text{LHS} = \text{RHS} \leqslant \cdots$.
- The first quantity we estimate is the RHS of (3.5),
- The next step is to estimate the LHS of (3.5), and we begin by estimating the viscous energy and the axial and radial displacements energy.
- After we have proved estimates for each part we can state the energy estimates for the whole system.
- The energy estimate will now guide us to determine the leading order behavior of an asymptotic expansion.

We have that $k(\varepsilon)G(\varepsilon) - \frac{\sigma}{1-\sigma^2}E(\varepsilon)$ is positive, and collect positive terms, we take care of the negative terms and then see which positive term we will need, so we do the following arrangement:

$$\omega^\varepsilon h(\varepsilon)\frac{d}{2dt}\Big\{(\omega^\varepsilon)^2\varepsilon R\rho_\omega\Big(\|\frac{\partial\eta^\varepsilon}{\partial t}\|^2_{L_2(0,L)}+\|\frac{\partial s^\varepsilon}{\partial t}\|^2_{L_2(0,L)}\Big)+\Big(k(\varepsilon)G(\varepsilon)-\frac{E(\varepsilon)\sigma}{1-\sigma^2}\Big)\varepsilon R\|\frac{\partial\eta^\varepsilon}{\partial z}\|^2_{L_2(0,L)}$$
$$+\frac{E(\varepsilon)\varepsilon R}{1-\sigma^2}\Big[\sigma\|\frac{\partial s^\varepsilon}{\partial z}+\frac{\eta^\varepsilon}{\varepsilon R}\|^2_{L_2(0,L)}+(1-\sigma)\Big(\|\frac{\eta^\varepsilon}{\varepsilon R}\|^2_{L_2(0,L)}+\|\frac{\partial s^\varepsilon}{\partial z}\|^2_{L_2(0,L)}\Big)+\sigma\|\frac{\partial\eta^\varepsilon}{\partial z}-\frac{s^\varepsilon}{\varepsilon R}\|^2_{L_2(0,L)}$$
$$-\sigma\|\frac{s^\varepsilon}{\varepsilon R}\|^2_{L_2(0,L)}\Big]\Big\}+2\mu\int_{\Omega_\varepsilon}\|D(v^\varepsilon)\|^2_F r\,dr\,dz-\frac{\varepsilon R}{L}\int_0^L A(t)zv_r^\varepsilon(t,\varepsilon R,z)dz$$
$$+\omega^\varepsilon\frac{\varepsilon^2R^2}{2}A(t)\frac{\partial}{\partial t}\int_0^L s^\varepsilon(t,z)dz=-\int_{\Omega_\varepsilon}\frac{A(t)}{L}v_z^\varepsilon r\,dr\,dz+\omega^\varepsilon\frac{\varepsilon^2R^2}{2}A(t)\frac{\partial}{\partial t}\int_0^L s^\varepsilon(t,z)dz \qquad (3.7)$$

First we estimate the RHS of (3.7) and this is variant of Biot Law which will relate the forcing term with the volume shear stress term in the viscous energy and the elastic energy of the membrane.

Lemma 3 *The following estimate holds:*

$$\Big|\int_{\Omega_\varepsilon}v_z^\varepsilon r\,dr\,dz-\omega^\varepsilon\frac{\varepsilon^2R^2}{2}\frac{\partial}{\partial t}\int_0^L s^\varepsilon(t,z)dz\Big|\leqslant\frac{R^2L^{\frac12}}{2}\varepsilon^2\Big\|\frac12\big(\frac{\partial v_z^\varepsilon}{\partial r}+\frac{\partial v_r^\varepsilon}{\partial z}\big)\Big\|_{L^2(\Omega_\varepsilon)}$$

Proof:

$$\int_{\Omega_\varepsilon}v_z^\varepsilon r\,dr\,dz$$
$$=\int_0^L\int_0^{\varepsilon R}v_z^\varepsilon d\big(\frac{r^2}{2}\big)dz$$
$$=\int_0^L v_z^\varepsilon(\varepsilon R,t,z)\frac{\varepsilon^2R^2}{2}dz-\int_0^L\int_0^{\varepsilon R}\frac{r^2}{2}\frac{\partial v_z^\varepsilon}{\partial r}dr\,dz$$
$$=\omega^\varepsilon\int_0^L\frac{\partial s^\varepsilon}{\partial t}\frac{\varepsilon^2R^2}{2}dz-\int_{\Omega_\varepsilon}\frac{r}{2}\big(\frac{\partial v_z^\varepsilon}{\partial r}\big)r\,dr\,dz$$
$$\Rightarrow\int_{\Omega_\varepsilon}v_z^\varepsilon r\,dr\,dz-\omega^\varepsilon\frac{\varepsilon^2R^2}{2}\frac{\partial}{\partial t}\int_0^L s^\varepsilon(t,z)dz=-\int_{\Omega_\varepsilon}\frac{r}{2}\big(\frac{\partial v_z^\varepsilon}{\partial r}\big)r\,dr\,dz$$
$$\Rightarrow\Big|\int_{\Omega_\varepsilon}v_z^\varepsilon r\,dr\,dz-\omega^\varepsilon\frac{\varepsilon^2R^2}{2}\frac{\partial}{\partial t}\int_0^L s^\varepsilon(t,z)dz\Big|=\Big|\int_{\Omega_\varepsilon}\frac{r}{2}\big(\frac{\partial v_z^\varepsilon}{\partial r}\big)r\,dr\,dz\Big|$$
$$\Rightarrow\Big|\int_{\Omega_\varepsilon}v_z^\varepsilon r\,dr\,dz-\omega^\varepsilon\frac{\varepsilon^2R^2}{2}\frac{\partial}{\partial t}\int_0^L s^\varepsilon(t,z)dz\Big|\leqslant\Big|\int_{\Omega_\varepsilon}\frac{r}{2}\big(\frac{\partial v_z^\varepsilon}{\partial r}+\frac{\partial v_r^\varepsilon}{\partial z}\big)r\,dr\,dz\Big|$$
$$\leqslant\Big(\int_{\Omega_\varepsilon}r^2r\,dr\,dz\Big)^{1/2}\Big(\int_{\Omega_\varepsilon}\big(\frac12\big(\frac{\partial v_z^\varepsilon}{\partial r}+\frac{\partial v_r^\varepsilon}{\partial z}\big)\big)^2r\,dr\,dz\Big)^{1/2}$$
$$=\frac{\varepsilon^2R^2}{2}L^{\frac12}\Big\|\frac12\big(\frac{\partial v_z^\varepsilon}{\partial r}+\frac{\partial v_r^\varepsilon}{\partial z}\big)\Big\|_{L^2(\Omega_\varepsilon)}. \qquad \blacksquare$$

The following estimate relates the viscous energy to the forcing term,

$$\frac{\varepsilon^2 R^2\sqrt{L}}{2}|A(t)|\left\|\frac{1}{2}(\frac{\partial v_z^\varepsilon}{\partial r}+\frac{\partial v_r^\varepsilon}{\partial z})\right\|_{L^2(\Omega_\varepsilon)} \leqslant 2\mu\left\|\frac{1}{2}(\frac{\partial v_z^\varepsilon}{\partial r}+\frac{\partial v_r^\varepsilon}{\partial z})\right\|^2_{L^2(\Omega_\varepsilon)} + \frac{R^4\varepsilon^4 L}{8\mu}|A(t)|^2,$$

Proof :

$$\frac{\varepsilon^2 R^2\sqrt{L}}{2}|A(t)|\left\|\frac{1}{2}(\frac{\partial v_z^\varepsilon}{\partial r}+\frac{\partial v_r^\varepsilon}{\partial z})\right\|_{L^2(\Omega_\varepsilon)} = \Big(\frac{\varepsilon^2 R^2\sqrt{L}}{2\sqrt{2\mu}}|A(t)|\Big)\Big(\sqrt{2\mu}\left\|\frac{1}{2}(\frac{\partial v_z^\varepsilon}{\partial r}+\frac{\partial v_r^\varepsilon}{\partial z})\right\|_{L^2(\Omega_\varepsilon)}\Big)$$

Then using *Cauchy-Schwarz Inequality*, we can get the result. ■

3.1 The viscous energy estimate

The viscous term in the equality, $2\mu\int_{\Omega_\varepsilon}\|D(v^\varepsilon)\|_F^2 rdrdz$, satisfies the following energy estimate:

$$2\mu\int_{\Omega_\varepsilon}\|D(v^\varepsilon)\|_F^2 rdrdz \geqslant 2\mu\left\|\frac{1}{2}(\frac{\partial v_z^\varepsilon}{\partial r}+\frac{\partial v_r^\varepsilon}{\partial z})\right\|^2_{L^2(\Omega_\varepsilon)} + \mu\int_{\Omega_\varepsilon}\|D(v^\varepsilon)\|_F^2 rdrdz$$

since

$$\begin{aligned}&\mu\int_{\Omega_\varepsilon}\|D(v^\varepsilon)\|_F^2 rdrdz\\ &= \mu\Big\{\left\|\frac{\partial v_r^\varepsilon}{\partial r}\right\|^2_{L^2(\Omega_\varepsilon)} + \left\|\frac{v_r^\varepsilon}{r}\right\|^2_{L^2(\Omega_\varepsilon)} + \left\|\frac{\partial v_z^\varepsilon}{\partial z}\right\|^2_{L^2(\Omega_\varepsilon)} + 2\left\|\frac{1}{2}(\frac{\partial v_z^\varepsilon}{\partial r}+\frac{\partial v_r^\varepsilon}{\partial z})\right\|^2_{L^2(\Omega_\varepsilon)}\Big\}\\ &\geqslant 2\mu\left\|\frac{1}{2}(\frac{\partial v_z^\varepsilon}{\partial r}+\frac{\partial v_r^\varepsilon}{\partial z})\right\|^2_{L^2(\Omega_\varepsilon)}\end{aligned}$$

Throughout the text we will use the following notation:

$$\|A(t)\|^2_{\mathcal{H}} = max_{0\leq\tau\leq t}|A(\tau)|^2 + \int_0^t |\partial_\tau A(\tau)|^2 d\tau$$

3.2 Estimate of Axial Displacement s^ε

An estimate for the axial component of the displacement in the energy equality is given by the following lemma.

Lemma 4 *The z derivative of the axial displacement,* $\frac{\partial s^\varepsilon}{\partial z}$*, satisfies the following estimate:*

$$\begin{aligned}&\left|\frac{h(\varepsilon)E(\varepsilon)\varepsilon R}{2(1+\sigma)}\left\|\frac{\partial s^\varepsilon}{\partial z}(t)\right\|^2_{L^2(0,L)} + \frac{\varepsilon^2 R^2}{2L}\int_0^t A(\tau)\big(\frac{\partial}{\partial\tau}\int_0^L s^\varepsilon(\tau,z)dz\big)d\tau\right|\\ &\geqslant \frac{h(\varepsilon)E(\varepsilon)\varepsilon R}{4(1+\sigma)}\Big(\left\|\frac{\partial s^\varepsilon}{\partial z}(t)\right\|^2_{L^2(0,L)} - \int_0^t\left\|\frac{\partial s^\varepsilon}{\partial z}(\tau)\right\|^2_{L^2(0,L)}d\tau\Big) - \frac{(1+\sigma)\varepsilon^3 R^3 L}{12h(\varepsilon)E(\varepsilon)}\|A(t)\|^2_{\mathcal{H}}.\end{aligned}$$

Proof: *First we have this equality*

$$\int_0^t A(\tau)\big(\frac{\partial}{\partial\tau}\int_0^L s^\varepsilon(\tau,z)dz\big)d\tau = A(t)\int_0^L zs^\varepsilon(t,z)dz - \int_0^t \big(\frac{\partial A(\tau)}{\partial\tau}\int_0^L s^\varepsilon(\tau,z)dz\big)d\tau$$

plug in this equality to the LHS, we get:

$$\begin{aligned}
&\left|\frac{h(\varepsilon)E(\varepsilon)\varepsilon R}{2(1+\sigma)}\left\|\frac{\partial s^\varepsilon}{\partial z}(t)\right\|^2_{L^2(0,L)} + \frac{\varepsilon^2R^2}{2L}\Big(A(t)\int_0^L s^\varepsilon(t,z)dz - \int_0^t \frac{\partial A(\tau)}{\partial\tau}\int_0^L s^\varepsilon(\tau,z)dzd\tau\Big)\right| \\
&= \left|\frac{h(\varepsilon)E(\varepsilon)\varepsilon R}{2(1+\sigma)}\Big(\left\|\frac{\partial s^\varepsilon}{\partial z}(t)\right\|^2_{L^2(0,L)} + \underbrace{\frac{(1+\sigma)\varepsilon R}{Lh(\varepsilon)E(\varepsilon)}A(t)\int_0^L s^\varepsilon(t,z)dz}_{\bigstar_1}\right. \\
&\left.- \underbrace{\frac{(1+\sigma)\varepsilon R}{Lh(\varepsilon)E(\varepsilon)}\int_0^t \frac{\partial A(\tau)}{\partial\tau}\int_0^L s^\varepsilon(\tau,z)dzd\tau}_{\bigstar_2}\Big)\right| \\
&\geqslant \frac{h(\varepsilon)E(\varepsilon)\varepsilon R}{2(1+\sigma)}\left\{\left\|\frac{\partial s^\varepsilon}{\partial z}(t)\right\|^2_{L^2(0,L)} - |\bigstar_1| - |\bigstar_2|\right\}
\end{aligned}$$

$$\begin{aligned}
|\bigstar_1| &\leqslant \frac{L(1+\sigma)^2\varepsilon^2R^2}{6h^2(\varepsilon)E^2(\varepsilon)} max_{0<\tau<t}|A(\tau)|^2 + \frac{1}{2}\left\|\frac{\partial s^\varepsilon}{\partial z}(t)\right\|^2_{L^2(0,L)} \\
|\bigstar_2| &\leqslant \frac{L(1+\sigma)^2\varepsilon^2R^2}{6h^2(\varepsilon)E^2(\varepsilon)}\int_0^t |\partial_\tau A(\tau)|^2 d\tau + \frac{1}{2}\int_0^t \left\|\frac{\partial s^\varepsilon}{\partial z}(t)\right\|^2_{L^2(0,L)} d\tau
\end{aligned}$$

■

Lemma 5 *The axial displacement, s^ε, satisfies the estimate:*

$$\begin{aligned}
&\left|\frac{h(\varepsilon)E(\varepsilon)\varepsilon R}{2(1-\sigma^2)}(-\sigma)\left\|\frac{s^\varepsilon}{\varepsilon R}\right\|^2_{L^2(0,L)} + \frac{\varepsilon^2R^2}{2L}\int_0^t A(\tau)\big(\frac{\partial}{\partial\tau}\int_0^L s^\varepsilon(\tau,z)dz\big)d\tau\right| \\
&\geqslant \frac{h(\varepsilon)E(\varepsilon)\sigma}{4(1-\sigma^2)\varepsilon R}\Big(\|s^\varepsilon\|^2_{L^2(0,L)} - \int_0^t \|s^\varepsilon\|^2_{L^2(0,L)}d\tau\Big) - \frac{(1-\sigma^2)\varepsilon^5R^5}{4Lh(\varepsilon)E(\varepsilon)\sigma}\|A(t)\|^2_{\mathcal{H}}.
\end{aligned}$$

Proof : *First we have this equality*

$$\int_0^t A(\tau)\big(\frac{\partial}{\partial\tau}\int_0^L s^\varepsilon(\tau,z)dz\big)d\tau = A(t)\int_0^L zs^\varepsilon(t,z)dz - \int_0^t \big(\frac{\partial A(\tau)}{\partial\tau}\int_0^L s^\varepsilon(\tau,z)dz\big)d\tau$$

by inserting this equality to the LHS, we get:

$$\Big|\frac{h(\varepsilon)E(\varepsilon)\varepsilon R}{2(1-\sigma^2)}(-\sigma)\Big\|\frac{s^\varepsilon}{\varepsilon R}\Big\|^2_{L^2(0,L)}+\frac{\varepsilon^2R^2}{2L}\Big(A(t)\int_0^L s^\varepsilon(t,z)dz-\int_0^t\frac{\partial A(\tau)}{\partial\tau}\int_0^L s^\varepsilon(\tau,z)dzd\tau\Big)\Big|$$

$$=\Big|\frac{h(\varepsilon)E(\varepsilon)\sigma}{2(1-\sigma^2)\varepsilon R}\Big(\|s^\varepsilon\|^2_{L^2(0,L)}-\underbrace{\frac{(1-\sigma^2)\varepsilon^3R^3}{Lh(\varepsilon)E(\varepsilon)\sigma}A(t)\int_0^L s^\varepsilon(t,z)dz}_{\bigstar_1}$$

$$+\underbrace{\frac{(1-\sigma^2)\varepsilon^3R^3}{Lh(\varepsilon)E(\varepsilon)\sigma}\int_0^t\frac{\partial A(\tau)}{\partial\tau}\int_0^L s^\varepsilon(\tau,z)dzd\tau}_{\bigstar_2}\Big)\Big|$$

$$\geqslant\frac{h(\varepsilon)E(\varepsilon)\sigma}{2(1-\sigma^2)\varepsilon R}\Big\{\|s^\varepsilon\|^2_{L^2(0,L)}-|\bigstar_1|-|\bigstar_2|\Big\}$$

$$|\bigstar_1|\leqslant\frac{(1-\sigma^2)^2\varepsilon^6R^6}{2Lh^2(\varepsilon)E^2(\varepsilon)\sigma^2}max_{0<\tau<t}|A(\tau)|^2+\frac{1}{2}\|s^\varepsilon\|^2_{L^2(0,L)}$$

$$|\bigstar_2|\leqslant\frac{(1-\sigma^2)^2\varepsilon^6R^6}{2Lh^2(\varepsilon)E^2(\varepsilon)\sigma^2}\int_0^t|\partial_\tau A(\tau)|^2d\tau+\frac{1}{2}\int_0^t\|s^\varepsilon\|^2_{L^2(0,L)}d\tau \qquad \blacksquare$$

3.3 Estimate of Radial Displacement η^ε

An estimate for the radial displacement is given by the following lemma:

Lemma 6 *The radial displacement, η^ε, satisfies the estimate:*

$$\Big|\frac{h(\varepsilon)E(\varepsilon)}{2\varepsilon R(1+\sigma)}\|\eta^\varepsilon(t)\|^2_{L^2(0,L)}-\frac{\varepsilon R}{L}\int_0^t A(\tau)\frac{\partial}{\partial\tau}\Big(\int_0^L z\eta^\varepsilon(\tau,z)dz\Big)d\tau\Big|$$

$$\geqslant\frac{h(\varepsilon)E(\varepsilon)}{4\varepsilon R(1+\sigma)}\Big(\Big\|\eta^\varepsilon(t)\Big\|^2_{L^2(0,L)}-\int_0^t\Big\|\eta^\varepsilon(\tau)\Big\|^2_{L^2(0,L)}d\tau\Big)-\frac{(1+\sigma)\varepsilon^3R^3L}{3h(\varepsilon)E(\varepsilon)}\|A(t)\|^2_{\mathcal{H}}.$$

Proof *: First we have the following equality:*

$$\int_0^t A(\tau)\frac{\partial}{\partial\tau}\Big(\int_0^L z\eta^\varepsilon(\tau,z)dz\Big)d\tau=A(t)\int_0^L z\eta^\varepsilon(t,z)dz-\int_0^t\Big(\frac{\partial A(\tau)}{\partial\tau}\int_0^L z\eta^\varepsilon(\tau,z)dz\Big)d\tau$$

we replace the second term by this equality, and obtain

$$\Big|\frac{h(\varepsilon)E(\varepsilon)}{2\varepsilon R(1+\sigma)}\|\eta^\varepsilon(t)\|^2_{L^2(0,L)}-\frac{\varepsilon R}{L}\Big(A(t)\int_0^L z\eta^\varepsilon(t,z)dz-\int_0^t\Big(\frac{\partial A(\tau)}{\partial\tau}\int_0^L z\eta^\varepsilon(\tau,z)dz\Big)d\tau\Big)$$

$$=\Big|\frac{h(\varepsilon)E(\varepsilon)}{2\varepsilon R(1+\sigma)}\Big(\|\eta^\varepsilon(t)\|^2_{L^2(0,L)}-\underbrace{\frac{2\varepsilon^2R^2(1+\sigma)}{Lh(\varepsilon)E(\varepsilon)}A(t)\int_0^L z\eta^\varepsilon(t,z)dz}_{\bigstar_3}$$

$$+\underbrace{\frac{2\varepsilon^2R^2(1+\sigma)}{Lh(\varepsilon)E(\varepsilon)}\int_0^t\Big(\frac{\partial A(\tau)}{\partial\tau}\int_0^L z\eta^\varepsilon(\tau,z)dz\Big)d\tau}_{\bigstar_4}\Big)\Big|$$

$$\geqslant \frac{h(\varepsilon)E(\varepsilon)}{2\varepsilon R(1+\sigma)}\left\{\left\|\eta^\varepsilon(t)\right\|^2_{L^2(0,L)} - |\bigstar_3| - |\bigstar_4|\right\}$$

$$|\bigstar_3| \leqslant \frac{2L(1+\sigma)^2\varepsilon^4 R^4}{3h^2(\varepsilon)E^2(\varepsilon)} max_{0<\tau<t}|A(\tau)|^2 + \frac{1}{2}\left\|\eta^\varepsilon\right\|^2_{L^2(0,L)}$$

$$|\bigstar_4| \leqslant\leqslant \frac{2L(1+\sigma)^2\varepsilon^4 R^4}{3h^2(\varepsilon)E^2(\varepsilon)} \int_0^t |\partial_\tau A(\tau)|^2 d\tau + \frac{1}{2}\int_0^t \left\|\eta^\varepsilon\right\|^2_{L^2(0,L)} d\tau \qquad \blacksquare$$

3.4 An energy estimate for the whole system

Lemma 7 *The radial displacement η^ε, the axial displacement s^ε, the viscous energy $\mu\int_{\Omega_\varepsilon}\|D(v^\varepsilon)\|^2_F rdrdz$, and the energy induced by the pressure drop $A(t)$ satisfy the following energy estimate*

$$\omega^\varepsilon \frac{h(\varepsilon)E(\varepsilon)\sigma}{4\varepsilon R(1-\sigma^2)}\left\{\left\|s^\varepsilon(t)\right\|^2_{L^2(0,L)} + \frac{1-\sigma}{\sigma}\left\|\eta^\varepsilon(t)\right\|^2_{L^2(0,L)} + \frac{1-\sigma}{\sigma}\varepsilon^2 R^2\left\|\frac{\partial s^\varepsilon}{\partial z}(t)\right\|^2_{L^2(0,L)}\right\}$$
$$+\mu\int_0^t\int_{\Omega_\varepsilon}\left\|D(v^\varepsilon)\right\|^2_F rdrdzd\tau$$
$$\leqslant \omega^\varepsilon \frac{h(\varepsilon)E(\varepsilon)\sigma}{4\varepsilon R(1-\sigma^2)}\int_0^t\left\{\left\|s^\varepsilon(t)\right\|^2_{L^2(0,L)} + \frac{1-\sigma}{\sigma}\left\|\eta^\varepsilon(t)\right\|^2_{L^2(0,L)} + \frac{1-\sigma}{\sigma}\varepsilon^2 R^2\left\|\frac{\partial s^\varepsilon}{\partial z}(t)\right\|^2_{L^2(0,L)}\right\}d\tau$$
$$+\frac{R^4\varepsilon^4 L}{16\mu}\int_0^t |A(t)|^2 d\tau + \omega^\varepsilon\left\{\frac{5(1+\sigma)\varepsilon^3 R^3 L}{12h(\varepsilon)E(\varepsilon)} + \frac{(1-\sigma^2)\varepsilon^5 R^5}{4Lh(\varepsilon)E(\varepsilon)\sigma}\right\}\|A(t)\|^2_{\mathcal{H}} \qquad (3.8)$$

By applying the Gronwall inequality to (3.8), we get an estimate which is crucial to determine the leading order behavior in asymptotic expansions.

Proposition 2 *The solution $(v_r^\varepsilon, v_z^\varepsilon, \eta^\varepsilon, s^\varepsilon)$ to (2.19) satisfies the estimate:*

$$\omega^\varepsilon \frac{h(\varepsilon)E(\varepsilon)\sigma}{4\varepsilon R(1-\sigma^2)}\left\{\left\|s^\varepsilon(t)\right\|^2_{L^2(0,L)} + \frac{1-\sigma}{\sigma}\left\|\eta^\varepsilon(t)\right\|^2_{L^2(0,L)} + \frac{1-\sigma}{\sigma}\varepsilon^2 R^2\left\|\frac{\partial s^\varepsilon}{\partial z}(t)\right\|^2_{L^2(0,L)}\right\}$$
$$+\mu\int_0^t\int_{\Omega_\varepsilon}\left\|D(v^\varepsilon)\right\|^2_F rdrdzd\tau$$
$$\leqslant \left\{\frac{R^4\varepsilon^4 L}{16\mu}\int_0^t |A(t)|^2 d\tau + \omega^\varepsilon\left(\frac{5(1+\sigma)\varepsilon^3 R^3 L}{12h(\varepsilon)E(\varepsilon)} + \frac{(1-\sigma^2)\varepsilon^5 R^5}{4Lh(\varepsilon)E(\varepsilon)\sigma}\right)\|A(t)\|^2_{\mathcal{H}}\right\}e^t \qquad (3.9)$$

To capture the elastic response of the membrane to the oscillations in the pressure drop between the inlet and the outlet boundary, ω^ε is chosen so that both terms on the right-hand side are of the same order in ε. We notice that the term which contains ε^3 determine the equation when ε is small. Using the assumptions in *Assumption 1*, we get

$$\omega^\varepsilon = \frac{\varepsilon^2}{\mu}$$

We are now ready to obtain the a priori estimates in terms of ε. In the following text, we denote all constants independent of ε by C. Define:

$$\|A\|_v^2 = e^T\Big(\|A\|^2_{L^2(0,L)} + \int_0^T \big(|\partial_\tau A(\tau)|^2 + |A(\tau)|^2\big) d\tau\Big)$$

and simplified the following notation

$$\big\|D(v^\varepsilon)\big\|_{L^2(\Omega_\varepsilon)} = \int_{\Omega_\varepsilon} \big\|D(v^\varepsilon)\big\|_F r dr dz$$

Proposition 3 *The solution $(v_r^\varepsilon, v_z^\varepsilon, \eta^\varepsilon, s^\varepsilon)$ to (2.19) satisfies the a priori estimates:*

$$\int_0^t \Big\{\Big\|\frac{\partial v_r^\varepsilon}{\partial r}\Big\|^2_{L^2(\Omega_\varepsilon)} + \Big\|\frac{v_r^\varepsilon}{r}\Big\|^2_{L^2(\Omega_\varepsilon)} + \Big\|\frac{\partial v_z^\varepsilon}{\partial z}\Big\|^2_{L^2(\Omega_\varepsilon)}\Big\} \leqslant C\Big(\frac{\varepsilon^2}{\mu}\Big)^2 \|A\|_v^2 \tag{3.10}$$

$$\Big\|\frac{\partial s^\varepsilon}{\partial z}(t)\Big\|^2_{L^2(0,L)} + \frac{1}{\varepsilon^2 R^2}\|\eta^\varepsilon(t)\|^2_{L^2(0,L)} \leqslant C\frac{\varepsilon}{h(\varepsilon)E(\varepsilon)}\|A\|_v^2 \tag{3.11}$$

$$\int_0^t \|v_z^\varepsilon\|^2_{L^2(\Omega_\varepsilon)} d\tau \leqslant \int_0^t \Big(\Big\|\frac{\partial v_z^\varepsilon}{\partial z}\Big\|^2_{L^2(\Omega_\varepsilon)} + \Big\|\frac{\partial v_z^\varepsilon}{\partial r}\Big\|^2_{L^2(\Omega_\varepsilon)}\Big) d\tau \tag{3.12}$$

$$\begin{aligned}\int_0^t \Big(\Big\|\frac{\partial v_r^\varepsilon}{\partial z}\Big\|^2_{L^2(\Omega_\varepsilon)} + \Big\|\frac{\partial v_z^\varepsilon}{\partial r}\Big\|^2_{L^2(\Omega_\varepsilon)}\Big) d\tau \leqslant 2C\big(\frac{\varepsilon^2}{\mu}\big)^2 \|A\|_v^2 + 2\int_0^t \Big\|\frac{\partial v_z^\varepsilon}{\partial z}\Big\|^2_{L^2(\Omega_\varepsilon)} d\tau \\ +(\omega^\varepsilon)^2 \int_0^t \|\partial_\tau s^\varepsilon(\tau)\|^2_{L^2(0,L)} d\tau + \varepsilon^2 R \int_0^t \Big\|\frac{\partial v_r^\varepsilon}{\partial z}(\varepsilon R, z, \tau)\Big\|^2_{L^2(0,L)} d\tau\end{aligned} \tag{3.13}$$

Proof :

1. *From (3.9), we have*

$$\mu \int_0^t \big\|D(v^\varepsilon)\big\|^2_{L^2(\Omega_\varepsilon)} d\tau \leqslant RHS(3.9) \leqslant C\Big(\frac{\varepsilon^2}{\mu}\Big)^2 \big\|A\big\|_v^2$$

$$\Big\|\frac{\partial v_r^\varepsilon}{\partial r}\Big\|^2_{L^2(\Omega_\varepsilon)} + \Big\|\frac{v_r^\varepsilon}{r}\Big\|^2_{L^2(\Omega_\varepsilon)} + \Big\|\frac{\partial v_z^\varepsilon}{\partial z}\Big\|^2_{L^2(\Omega_\varepsilon)} + 2\Big\|\frac{1}{2}\big(\frac{\partial v_r^\varepsilon}{\partial z} + \frac{\partial v_z^\varepsilon}{\partial r}\big)\Big\|^2_{L^2(\Omega_\varepsilon)} = \big\|D(v^\varepsilon)\big\|^2_{L^2(\Omega_\varepsilon)}$$

2. *Also from (3.9), we have*

$$\omega^\varepsilon \frac{h(\varepsilon)E(\varepsilon)\sigma}{4\varepsilon R(1-\sigma^2)}\Big\{\frac{1-\sigma}{\sigma}\big\|\eta^\varepsilon(t)\big\|^2_{L^2(0,L)} + \frac{1-\sigma}{\sigma}\varepsilon^2 R^2 \Big\|\frac{\partial s^\varepsilon}{\partial z}(t)\Big\|^2_{L^2(0,L)}\Big\} \leqslant C\Big(\frac{\varepsilon^2}{\mu}\Big)^2 \big\|A\big\|_v^2$$

$$\Rightarrow \big\|\eta^\varepsilon(t)\big\|^2_{L^2(0,L)} + \varepsilon^2 R^2 \Big\|\frac{\partial s^\varepsilon}{\partial z}(t)\Big\|^2_{L^2(0,L)} \leqslant C\frac{\varepsilon}{h(\varepsilon)E(\varepsilon)}\big\|A\big\|_v^2$$

3. *The third one comes from **Poincaré Inequality**.*

4. From the proof of the first inequality, we have

$$\int_0^t 2\Big\|\frac{1}{2}(\frac{\partial v_r^\varepsilon}{\partial z}+\frac{\partial v_z^\varepsilon}{\partial r})\Big\|^2_{L^2(\Omega_\varepsilon)}d\tau \leqslant \int_0^t \big\|D(v^\varepsilon)\big\|^2_{L^2(\Omega_\varepsilon)}d\tau \leqslant C\Big(\frac{\varepsilon^2}{\mu}\Big)^2\big\|A\big\|_v^2$$

then we get:

$$\int_0^t\int_0^L\int_0^{\varepsilon R}\bigg(\Big(\frac{\partial v_r^\varepsilon}{\partial z}\Big)^2+2\frac{\partial v_r^\varepsilon}{\partial z}\frac{\partial v_z^\varepsilon}{\partial r}+\Big(\frac{\partial v_z^\varepsilon}{\partial r}\Big)^2\bigg)rdrdzd\tau \leqslant 2C\Big(\frac{\varepsilon^2}{\mu}\Big)^2\big\|A\big\|_v^2$$

The difficulties come from the term which is the product of two off-diagonal gradient terms $\frac{\partial v_r^\varepsilon}{\partial z}\frac{\partial v_z^\varepsilon}{\partial r}$. *Estimate this term by using the boundary behavior of* v^ε,$\partial_z v_z^\varepsilon=0$ *at* $z=0,L$ *and we have:*

$$\begin{aligned}
&\int_0^L\int_0^{\varepsilon R}\frac{\partial v_r^\varepsilon}{\partial z}\frac{\partial v_z^\varepsilon}{\partial r}rdrdz\\
&=\int_0^L\int_0^{\varepsilon R}\frac{\partial v_r^\varepsilon}{\partial z}rdv_z^\varepsilon dz\\
&=\int_0^L\bigg(\Big(\frac{\partial v_r^\varepsilon}{\partial z}rv_z^\varepsilon\Big)\Big|_0^{\varepsilon R}-\int_0^{\varepsilon R}v_z^\varepsilon\frac{\partial}{\partial r}\Big(\frac{\partial v_r^\varepsilon}{\partial z}r\Big)dr\bigg)dz\\
&=\int_0^L\frac{\partial v_r^\varepsilon}{\partial z}(\varepsilon R,z)\varepsilon R v_z^\varepsilon(\varepsilon R,z)dz-\underbrace{\int_0^L\int_0^{\varepsilon R}v_z^\varepsilon\frac{\partial}{\partial r}\Big(\frac{\partial v_r^\varepsilon}{\partial z}r\Big)drdz}_{\bigstar 1}\\
&=\int_0^L\Big(\frac{\partial v_r^\varepsilon}{\partial z}r\Big)\Big|_0^{\varepsilon R}\omega^\varepsilon\frac{\partial s^\varepsilon}{\partial t}dz-\int_0^L\int_0^{\varepsilon R}\Big(\frac{\partial v_z^\varepsilon}{\partial z}\Big)^2rdrdz
\end{aligned}$$

we have used the relation in (2.11), $v_z^\varepsilon(\varepsilon R,z)=\frac{\partial s^\varepsilon}{\partial t}$ *and since*

$$\begin{aligned}
\bigstar 1 \quad &=\int_0^L\int_0^{\varepsilon R}v_z^\varepsilon\Big(\frac{\partial v_r^\varepsilon}{\partial z}+r\frac{\partial}{\partial z}\underbrace{(-\frac{v_r^\varepsilon}{r}-\frac{\partial v_z^\varepsilon}{\partial z})}_{using(2.9)}\Big)drdz\\
&=-\int_0^L\int_0^{\varepsilon R}v_z^\varepsilon\frac{\partial^2 v_z^\varepsilon}{\partial z^2}rdrdz\\
&=-\int_0^{\varepsilon R}\Big(v_z^\varepsilon\frac{\partial v_z^\varepsilon}{\partial z}\Big)\Big|_0^L rdr+\int_0^L\int_0^{\varepsilon R}\Big(\frac{\partial v_z^\varepsilon}{\partial z}\Big)^2rdrdz
\end{aligned}$$

now we can add this result and get:

$$\begin{aligned}
&\int_0^t\bigg(\Big\|\frac{\partial v_r^\varepsilon}{\partial z}\Big\|^2_{L^2(\Omega_\varepsilon)}+\Big\|\frac{\partial v_z^\varepsilon}{\partial r}\Big\|^2_{L^2(\Omega_\varepsilon)}\bigg)d\tau\\
&\leqslant 2\int_0^t\Big\|\frac{\partial v_z^\varepsilon}{\partial z}\Big\|^2_{L^2(\Omega_\varepsilon)}d\tau+2C\Big(\frac{\varepsilon^2}{\mu}\Big)^2\big\|A\big\|_v^2-2\int_0^t\int_0^L\Big(\frac{\partial v_r^\varepsilon}{\partial z}r\Big)\Big|_0^{\varepsilon R}\omega^\varepsilon\frac{\partial s^\varepsilon}{\partial t}dzd\tau
\end{aligned}$$

For the last term in the above inequality, we have used the Cauchy-Schwarz inequality *and the result follows.* ■

From (3.12) of **Proposition 3**, we can get the estimate for v_z^ε , since

$$\left\| \frac{\partial v_z^\varepsilon}{\partial z} \right\|_{L^2(\Omega_\varepsilon \times (0,T))} \leqslant C \frac{\varepsilon^2}{\mu} \|A\|_v$$
$$\left\| \frac{\partial v_z^\varepsilon}{\partial r} \right\|_{L^2(\Omega_\varepsilon \times (0,T))} \leqslant C \frac{\varepsilon^2}{\mu} \|A\|_v$$

then we add these two to (3.12), and we get the estimate for v_z^ε

$$\|v_z^\varepsilon\|_{L^2(\Omega_\varepsilon \times (0,T))} \leqslant C \frac{\varepsilon^2}{\mu} \|A\|_v$$

and always have this relationship in mind:

$$v^\varepsilon = v_r^\varepsilon \vec{e}_r + v_z^\varepsilon \vec{e}_z, \qquad \|v^\varepsilon\|^2 = \|v_r^\varepsilon\|^2 + \|v_z^\varepsilon\|^2$$

Now it is time for us to summarize the most important estimates in the following theorem. Here we recall the asymptotic behaviour ($\varepsilon \to 0$) of assumptions (2.4), (2.5) and (2.6) and use the same notation, E_0 and G_0 to denote the expressions $\frac{E(\varepsilon)h(\varepsilon)}{\varepsilon}$ and $G(\varepsilon)k(\varepsilon)h(\varepsilon)\varepsilon$ respectively. Thses estimates will now be used in the determination of the asymptotic expansions.

Theorem 2 *The solution $(v_r^\varepsilon, v_z^\varepsilon, \eta^\varepsilon, s^\varepsilon)$ to (2.19) satisfies the a priori estimates:*

$$\frac{1}{\varepsilon}\|\eta^\varepsilon(t)\|_{L^2(0,L)} \leqslant C\|A\|_v \tag{3.14}$$

$$\frac{1}{\varepsilon}\|s^\varepsilon(t)\|_{L^2(0,L)} \leqslant C\|A\|_v \tag{3.15}$$

$$\Big\|\frac{\partial s^\varepsilon}{\partial z}(t)\Big\|_{L^2(0,L)} \leqslant C\|A\|_v \tag{3.16}$$

$$\frac{1}{\varepsilon}\Big\|\frac{\partial \eta^\varepsilon}{\partial z}(t)\Big\|_{L^2(0,L)} \leqslant C\|A\|_v \tag{3.17}$$

$$\Big\|\frac{\partial v_z^\varepsilon}{\partial r}\Big\|_{L^2(\Omega_\varepsilon\times(0,T))} \leqslant C\frac{\varepsilon^2}{\mu}\|A\|_v \tag{3.18}$$

$$\Big\|\frac{\partial v_r^\varepsilon}{\partial r}\Big\|_{L^2(\Omega_\varepsilon\times(0,T))} \leqslant C\frac{\varepsilon^2}{\mu}\|A\|_v \tag{3.19}$$

$$\Big\|\frac{\partial v^\varepsilon}{\partial r}\Big\|_{L^2(\Omega_\varepsilon\times(0,T))} \leqslant C\frac{\varepsilon^2}{\mu}\|A\|_v \tag{3.20}$$

$$\Big\|\frac{\partial v_z^\varepsilon}{\partial z}\Big\|_{L^2(\Omega_\varepsilon\times(0,T))} \leqslant C\frac{\varepsilon^2}{\mu}\|A\|_v \tag{3.21}$$

$$\Big\|\frac{\partial v_r^\varepsilon}{\partial z}\Big\|_{L^2(\Omega_\varepsilon\times(0,T))} \leqslant C\frac{\varepsilon^2}{\mu}\|A\|_v \tag{3.22}$$

$$\Big\|\frac{\partial v^\varepsilon}{\partial z}\Big\|_{L^2(\Omega_\varepsilon\times(0,T))} \leqslant C\frac{\varepsilon^2}{\mu}\|A\|_v \tag{3.23}$$

$$\|v_r^\varepsilon\|_{L^2(\Omega_\varepsilon\times(0,T))} \leqslant C\frac{\varepsilon^2}{\mu}\|A\|_v \tag{3.24}$$

$$\|v_z^\varepsilon\|_{L^2(\Omega_\varepsilon\times(0,T))} \leqslant C\frac{\varepsilon^2}{\mu}\|A\|_v \tag{3.25}$$

$$\|v^\varepsilon\|_{L^2(\Omega_\varepsilon\times(0,T))} \leqslant C\frac{\varepsilon^2}{\mu}\|A\|_v \tag{3.26}$$

Chapter 4

The rescaled problem and asymptotic expansions

4.1 The rescaled problem

In order to study problem P^ε when $\varepsilon \to 0$, it is more convenient to use a rescaling which maps domain Ω_ε to a fixed domain $\Omega = \Omega_1$ corresponding to $\varepsilon = 1$. This, in turn, rescales the variables and their derivatives in the following manner.

$$\Omega_\varepsilon \to \Omega_1 \quad \varepsilon = 1 \tag{4.1}$$

$$v^\varepsilon \to v(\varepsilon) \tag{4.2}$$

$$v^\varepsilon = v_r^\varepsilon \vec{e}_r + v_z^\varepsilon \vec{e}_z \to v(\varepsilon) = v(\varepsilon)_r \vec{e}_r + v(\varepsilon)_z \vec{e}_z \tag{4.3}$$

Define new variable: $(y, z) := (\varepsilon r, z), y = \varepsilon r \Rightarrow r = \frac{1}{\varepsilon} y$,
New scaled functions: $v(\varepsilon)(r, z) = v^\varepsilon(\varepsilon r, z) = v^\varepsilon(y, z)$
Derivative: $\frac{\partial}{\partial y}(v^\varepsilon(y, z)) = \frac{\partial}{\partial r}(v^\varepsilon(\varepsilon r, z))\frac{\partial r}{\partial y} = \frac{1}{\varepsilon}\frac{\partial}{\partial r}(v(\varepsilon)(r, z))$

Let $v(\varepsilon)$ be an axially symmetric function defined in $\Omega_\varepsilon \in \mathbb{R}^3$, $v(\varepsilon) = v_r(\varepsilon)\vec{e}_r + v_z(\varepsilon)\vec{e}_z$, $\Omega \equiv \Omega_1$

The rescaled incompressible Stokes' equations (2.7), (2.8) and (2.9), defined on $\Omega \times \mathbb{R}_+$, read

$$-\mu\Big(\frac{1}{\varepsilon^2}\frac{\partial^2 v(\varepsilon)_r}{\partial r^2} + \frac{\partial^2 v(\varepsilon)_r}{\partial z^2} + \frac{1}{\varepsilon^2}\frac{1}{r}\frac{\partial v(\varepsilon)_r}{\partial r} - \frac{1}{\varepsilon^2}\frac{v(\varepsilon)_r}{r^2}\Big) + \frac{1}{\varepsilon}\frac{\partial p(\varepsilon)}{\partial r} = 0 \tag{4.4}$$

$$-\mu\Big(\frac{1}{\varepsilon^2}\frac{\partial^2 v(\varepsilon)_z}{\partial r^2} + \frac{\partial^2 v(\varepsilon)_z}{\partial z^2} + \frac{1}{\varepsilon^2}\frac{1}{r}\frac{\partial v(\varepsilon)_z}{\partial r}\Big) + \frac{\partial p(\varepsilon)}{\partial z} = 0 \tag{4.5}$$

$$div_\varepsilon v(\varepsilon) = \frac{1}{\varepsilon}\frac{\partial v(\varepsilon)_r}{\partial r} + \frac{\partial v(\varepsilon)_z}{\partial z} + \frac{1}{\varepsilon}\frac{v(\varepsilon)_r}{r} = 0 \tag{4.6}$$

since the quantities defined on the lateral boundary are invariant under this scaling, we use the same notation for the wall displacements of the rescaled problem as for the original problem, namely, η^ε and s^ε. The lateral wall's motion is described in Lagrangian coordinates by the Navier equations, and recall that we have rescaled the time, and still we use the same symbol t, we have

$$F_r = -\frac{h(\varepsilon)E(\varepsilon)}{1-\sigma^2}\Big(\frac{\sigma}{\varepsilon R}\frac{\partial s^\varepsilon}{\partial z} + \frac{\eta^\varepsilon}{\varepsilon^2 R^2}\Big) + h(\varepsilon)G(\varepsilon)k(\varepsilon)\frac{\partial^2 \eta^\varepsilon}{\partial z^2} - \rho_\omega h(\varepsilon)\frac{\varepsilon^4}{\mu^2}\frac{\partial^2 \eta^\varepsilon}{\partial t^2}, \tag{4.7}$$

$$F_z = \frac{h(\varepsilon)E(\varepsilon)}{1-\sigma^2}\Big(\frac{\partial^2 s^\varepsilon}{\partial z^2} + \frac{\sigma}{\varepsilon R}\frac{\partial \eta^\varepsilon}{\partial z}\Big) - \rho_\omega h(\varepsilon)\frac{\varepsilon^4}{\mu^2}\frac{\partial^2 s^\varepsilon}{\partial t^2}, \tag{4.8}$$

The lateral boundary conditions read:

$$v(\varepsilon)_r = \frac{\varepsilon^2}{\mu}\frac{\partial \eta^\varepsilon}{\partial t} \quad on \quad \Sigma \times \mathbb{R}_+ \tag{4.9}$$

$$v(\varepsilon)_z = \frac{\varepsilon^2}{\mu}\frac{\partial s^\varepsilon}{\partial t} \quad on \quad \Sigma \times \mathbb{R}_+ \tag{4.10}$$

$$-F_r = p(\varepsilon) - 2\mu\frac{1}{\varepsilon}\frac{\partial v(\varepsilon)_r}{\partial r} \quad on \quad \Sigma \times \mathbb{R}_+ \tag{4.11}$$

$$-F_z = -\mu(\frac{\partial v(\varepsilon)_r}{\partial z} + \frac{1}{\varepsilon}\frac{\partial v(\varepsilon)_z}{\partial r}) \quad on \quad \Sigma \times \mathbb{R}_+ \tag{4.12}$$

where $D_\varepsilon(v(\varepsilon)) = \frac{1}{2}(\nabla v(\varepsilon) + (\nabla v(\varepsilon))^T)$. i.e. if $\psi = \psi_r \vec{e}_r + \psi_z \vec{e}_z$,

$$D_\varepsilon(\psi) = \begin{pmatrix} \frac{1}{\varepsilon}\frac{\partial \psi_r}{\partial r} & 0 & \frac{1}{2}\left(\frac{\partial \psi_r}{\partial z} + \frac{1}{\varepsilon}\frac{\partial \psi_z}{\partial r}\right) \\ 0 & \frac{1}{\varepsilon}\frac{\psi_r}{r} & 0 \\ \frac{1}{2}\left(\frac{\partial \psi_r}{\partial z} + \frac{1}{\varepsilon}\frac{\partial \psi_z}{\partial r}\right) & 0 & \frac{\partial \psi_z}{\partial z} \end{pmatrix}$$

The initial conditions read:

$$\eta^\varepsilon = s^\varepsilon = \frac{\partial \eta^\varepsilon}{\partial t} = \frac{\partial s^\varepsilon}{\partial t} = 0 \quad on \quad \Sigma \times \{0\}. \tag{4.13}$$

The boundary data :

$$v(\varepsilon)_r = 0 \quad and \quad p(\varepsilon) = 0 \quad on \quad (\partial\Omega \cap \{z=0\}) \times \mathbb{R}_+, \tag{4.14}$$

$$v(\varepsilon)_r = 0 \quad and \quad p(\varepsilon) = A(t) \quad on \quad (\partial\Omega \cap \{z=L\}) \times \mathbb{R}_+, \tag{4.15}$$

$$\frac{\partial s^\varepsilon}{\partial z} = \eta^\varepsilon = 0 \; for \; z = 0, \quad s^\varepsilon = \eta^\varepsilon = 0 \; for \; z = L \quad and \quad \forall t \in \mathbb{R}_+. \tag{4.16}$$

4.1.1 Weak formulation

We define the test function space V and the solution space $\mathcal{V}$ as following.

Definition 4 *The space $V \subset H^1(\Omega)^3$ consists of all axially symmetric functions φ such that $\varphi_r|_\Sigma, \varphi_z|_\Sigma \in H^1(0,L)$, $\varphi_r(0,r) = \varphi_r(L,r) = 0, \varphi_z(L,R) = 0$ for $r \leq R$ and $div_\varepsilon \varphi = 0$ in Ω.*

Definition 5 *The space $\mathcal{V}$ consists of all functions $(w_r, w_z, d_r, d_z) \in H^1((0,T);V) \times (H^1((0,L)\times(0,T))^2 \bigcap H^2((0,T);L^2(0,L))^2)$ such that*

1. $\frac{1}{\varepsilon}\frac{\partial w_r}{\partial r} + \frac{\partial w_z}{\partial z} + \frac{1}{\varepsilon}\frac{w_r}{r} = 0$ *in* $\Omega \times \mathbb{R}_+$
2. $(\varepsilon r)^{-1} w_r \in L^2((0,T)\times\Omega)$
3. $d_r(t,0) = d_z(t,L) = d_r(t,L) = 0$ *on* $\mathbb{R}_+$
4. $w_r = 0$ *on* $(\partial\Omega \bigcap \{z=0\}) \times \mathbb{R}_+$
5. $w_r = 0$ *on* $(\partial\Omega \bigcap \{z=L\}) \times \mathbb{R}_+$
6. $w_r = \frac{\varepsilon^2}{\mu}\frac{\partial d_r}{\partial t}$ *and* $w_z = \frac{\varepsilon^2}{\mu}\frac{\partial d_z}{\partial t}$ *on* $\Sigma \times \mathbb{R}_+$.

Then rewriting variational equality (2.19) in rescaled variables, multiplying (2.19) by $\psi(t)$, and integrate on $\Omega \times (0,T)$ we obtain the variational formulation of the rescaled problem

$$\begin{aligned}
\boldsymbol{\varepsilon}^1_{elastic}(\eta^\varepsilon, s^\varepsilon, \varphi, \psi, \varepsilon) \;\equiv\;& R\int_0^T\int_0^L \Big\{ h(\varepsilon)G(\varepsilon)k(\varepsilon)\frac{\partial \eta^\varepsilon}{\partial z}\frac{\partial \varphi_r}{\partial z} + \frac{h(\varepsilon)E(\varepsilon)}{1-\sigma^2}\Big(\frac{\sigma}{\varepsilon R}\frac{\partial s^\varepsilon}{\partial z} \\
&+ \frac{\eta^\varepsilon}{\varepsilon^2 R^2}\Big)\varphi_r + \frac{h(\varepsilon)E(\varepsilon)}{1-\sigma^2}\Big(\frac{\partial s^\varepsilon}{\partial z}\frac{\partial \varphi_z}{\partial z} - \frac{\sigma}{\varepsilon R}\frac{\partial \eta^\varepsilon}{\partial z}\varphi_z\Big)\Big\}\Big|_{r=R}\psi(t)dzdt \\
&+ R\rho_\omega h(\varepsilon)\frac{\varepsilon^4}{\mu^2}\int_0^T \frac{d^2\psi(t)}{dt^2}\int_0^L (\eta^\varepsilon\varphi_r + s^\varepsilon\varphi_z)|_{r=R}\psi(t)dzdt \\
\boldsymbol{\varepsilon}^1_{pressure}(A, \varphi, \psi) \;\equiv\;& \int_0^T\int_0^R A(t)\varphi_z\psi(t)rdrdt \\
\boldsymbol{\varepsilon}^1_{fluid}(v(\varepsilon), \varphi, \psi; \varepsilon) \;\equiv\;& 2\mu\int_0^T\int_\Omega [D_\varepsilon(v(\varepsilon)) : D_\varepsilon(\varphi)]\psi(t)rdrdzdt
\end{aligned}$$

Definition 6 *(Weak formulation of the rescaled problem $P(\varepsilon)$) $(v(\varepsilon)_r, v(\varepsilon)_z, \eta^\varepsilon, s^\varepsilon) \in V$ is a weak solution of problem $P(\varepsilon)$ if the following variational formulation is satisfied:*

$$\boldsymbol{\varepsilon}^1_{fluid}(v(\varepsilon), \varphi, \psi; \varepsilon) + \boldsymbol{\varepsilon}^1_{elastic}(\eta^\varepsilon, s^\varepsilon, \varphi, \psi; \varepsilon) = -\boldsymbol{\varepsilon}^1_{pressure}(A, \varphi, \psi), \forall \varphi \in V, \forall \psi \in C(\mathbb{R}_+)$$

The initial conditions at the lateral boundary are:

$$\eta^\varepsilon = s^\varepsilon = \frac{\partial \eta^\varepsilon}{\partial t} = \frac{\partial s^\varepsilon}{\partial t} = 0 \qquad on \qquad \Sigma \times \{0\}$$

Later we will also need the weak formulation which includes the pressure. For this purpose we consider the test functions φ which is not divergence-free. Namely denoted by

$$\begin{aligned} V_{div_\varepsilon v(\varepsilon)\neq 0} = \ & \{\varphi \in H^1(\Omega)^3|_\varphi \ is \ axial \ symmetry, \varphi_r|_\Sigma, \varphi_z|_\Sigma \in H^1(0,L), \\ & \varphi_z(L,R) = \varphi_r(L,r) = \varphi_r(0,r) = 0\} \end{aligned}$$

Then the weak formulation of the problem, cast in terms of the velocity and pressure, reads as follows.

Definition 7 *(Weak Formulation of P(ε) in the pressure-velocity form) Vector function* $(v(\varepsilon)_r, v(\varepsilon)_z, \eta^\varepsilon, s^\varepsilon) \in \mathcal{V}$ *and* $p(\varepsilon) \in L^2(\Omega \times (0,T))$ *form a weak solution of problem* $P(\varepsilon)$ *if*

$$\begin{aligned} & \boldsymbol{\varepsilon}^1_{fluid}(v(\varepsilon), \varphi, \psi; \varepsilon) - \int_0^T \int_\Omega p(\varepsilon)(\frac{\partial \varphi_z}{\partial z} + \frac{1}{\varepsilon}\frac{\partial \varphi_r}{\partial r} + \frac{1}{\varepsilon}\frac{\varphi_r}{r})\psi(t) r dr dz dt \\ & + \boldsymbol{\varepsilon}^1_{elastic}(\eta^\varepsilon, s^\varepsilon, \varphi, \psi; \varepsilon) = -\boldsymbol{\varepsilon}^1_{pressure}(A, \varphi, \psi), \forall \varphi \in V_{div_\varepsilon v(\varepsilon)\neq 0} \ and \ \forall \psi \in C(\mathbb{R}_+) \end{aligned}$$

4.1.2 Energy estimates

Now we start to derive the energy estimates for the rescaled problem, which are crucial to determine the asymptotic expansions. We can now use the estimates from the previous section and we observe that we only need to change the terms which are rescaled. We have the following changes which inherit from the **Proposition 3**, write down those which are very important for the future use.

$$\Big(\Big\|\frac{1}{\varepsilon}\frac{v(\varepsilon)_r}{r}\Big\| + \Big\|\frac{1}{\varepsilon}\frac{\partial v(\varepsilon)_r}{\partial r}\Big\| + \Big\|\frac{\partial v(\varepsilon)_z}{\partial z}\Big\|\Big)_{L^2(\Omega\times(0,T))} \leqslant C\frac{\varepsilon^2}{\mu}\|A\|_v \tag{4.17}$$

From (4.17), we get the estimate for $v(\varepsilon)_r$,

$$\|v(\varepsilon)_r\|_{L^2(\Omega\times(0,T))} \leqslant C\frac{\varepsilon^3}{\mu}\|A\|_v$$

and (4.17) is the only equation we can find to estimate $v(\varepsilon)_r$ more precisely, even if we can use *Poincaré Inequality* to estimate it just like what we have done for $v(\varepsilon)_z$. The other estimates are straightforward consequences from **Proposition 3**. We just have to compensate for the rescaling in those terms which have derivative of r, we state these ones:

$$\Big\|\frac{\partial v(\varepsilon)_r}{\partial r}\Big\|_{L^2(\Omega\times(0,T))} \leqslant C\frac{\varepsilon^3}{\mu}\|A\|_v \tag{4.18}$$

$$\Big\|\frac{\partial v(\varepsilon)_z}{\partial r}\Big\|_{L^2(\Omega\times(0,T))} \leqslant C\frac{\varepsilon^3}{\mu}\|A\|_v \tag{4.19}$$

$$\Big\|\frac{\partial v(\varepsilon)}{\partial r}\Big\|_{L^2(\Omega\times(0,T))} \leqslant C\frac{\varepsilon^3}{\mu}\|A\|_v \tag{4.20}$$

Since now we have the estimates for $v(\varepsilon)_r$ and $v(\varepsilon)_z$, and recall $v(\varepsilon) = v(\varepsilon)_r \vec{e}_r + v(\varepsilon)_z \vec{e}_z$, then we have

$$\|v(\varepsilon)\|_{L^2(\Omega\times(0,T))} \leqslant C\frac{\varepsilon^2}{\mu}\|A\|_v$$

Here we summarize the results which we will use for the asymptotic expansions in order to determine the leading order terms.

Theorem 3 *The solution* $(v(\varepsilon)_r, v(\varepsilon)_z, p(\varepsilon), \eta^\varepsilon, s^\varepsilon)$ *of the rescaled problem as stated in* ***Definition 6*** *satisfies the following estimates*

$$\|v(\varepsilon)_r\|_{L^2(\Omega\times(0,T))} \leqslant C\frac{\varepsilon^3}{\mu}\|A\|_v \tag{4.21}$$

$$\|v(\varepsilon)_z\|_{L^2(\Omega\times(0,T))} \leqslant C\frac{\varepsilon^2}{\mu}\|A\|_v \tag{4.22}$$

$$\|v(\varepsilon)\|_{L^2(\Omega\times(0,T))} \leqslant C\frac{\varepsilon^2}{\mu}\|A\|_v \tag{4.23}$$

$$\|p(\varepsilon)\|_{L^2(\Omega\times(0,T))} \leqslant C\|A\|_v \tag{4.24}$$

$$\|\eta^\varepsilon(t)\|_{L^2(0,L)} \leqslant C\varepsilon\|A\|_v \tag{4.25}$$

$$\|s^\varepsilon(t)\|_{L^2(0,L)} \leqslant C\varepsilon\|A\|_v \tag{4.26}$$

4.2 Asymptotic expansions

Since now we have uniform estimates for $(v(\varepsilon)_r, v(\varepsilon)_z, p(\varepsilon), \eta^\varepsilon, s^\varepsilon)$, which are valid for their time derivatives as well, here we call this vector X^ε, i.e. $X^\varepsilon = (v(\varepsilon)_r, v(\varepsilon)_z, p(\varepsilon), \eta^\varepsilon, s^\varepsilon)$ which will be specified later in the chapter of convergence theorem. We can define the correct asymptotic expansions for X^ε. The usual difficulty is to determine the leading order term of ε. In general, they follow from the a priori estimates, but here we did not follow the a priori estimates strictly to determine the leading order term for s^ε. We have seen $\eta^\varepsilon, s^\varepsilon$ have the same estimate, but we observe that in our Navier equations for the elastic curved membrane, if we set the same leading order term, it will not lead our goal and they do not match each other, so we use the following a priori estimate for s^ε since $0 < \varepsilon < 1$, we have the following estimate from (4.26):

$$\|s^\varepsilon(t)\|_{L^2(0,L)} \leqslant C\|A\|_v$$

Asymptotic expansions I

$$v(\varepsilon)(z,r,t) = \frac{\varepsilon^2}{\mu}\sum_{i\geq 0}\varepsilon^i v^i(z,r,t), \tag{4.27}$$

$$p(\varepsilon)(z,r,t) = \sum_{i\geq 0}\varepsilon^i p^i(z,r,t), \tag{4.28}$$

$$\eta^\varepsilon(z,t) = \varepsilon\sum_{i\geq 0}\varepsilon^i \eta^i(z,t), \tag{4.29}$$

$$s^\varepsilon(z,t) = \sum_{i\geq 0}\varepsilon^i s^i(z,t), \tag{4.30}$$

and we have another asymptotic expansion which is also feasible and more compatible to the energy estimates we have used.

Asymptotic expansions II

$$v(\varepsilon)_r(z,r,t) = \frac{\varepsilon^3}{\mu}\sum_{i\geq 0}\varepsilon^i v_r^i(z,r,t), \tag{4.31}$$

$$v(\varepsilon)_z(z,r,t) = \frac{\varepsilon^2}{\mu}\sum_{i\geq 0}\varepsilon^i v_z^i(z,r,t), \tag{4.32}$$

$$p(\varepsilon)(z,r,t) = \sum_{i\geq 0}\varepsilon^i p^i(z,r,t), \tag{4.33}$$

$$\eta^\varepsilon(z,t) = \varepsilon\sum_{i\geq 0}\varepsilon^i \eta^i(z,t), \tag{4.34}$$

$$s^\varepsilon(z,t) = \sum_{i\geq 0}\varepsilon^i s^i(z,t), \tag{4.35}$$

In the following chapter we will show the two ways to get the reduced problems, the differences between them are $v(\varepsilon)_r$. But both ways lead to our goal. And I feel **Asymptotic expansions II** is better than **Asymptotic expansions I** no matter from what point.

Chapter 5

The reduced problem

5.1 The reduced problem P

5.1.1 Using the *Asymptotic Expansions I* to get the reduced equations of Stokes equations

In this section, we will use asymptotic expansions to derive the reduced problem. As we will see it will be second-order accurate in ε.

We derive the effective (reduced) equations, second-order accurate in ε, which hold for small ε.

• Insert (4.27) and (4.28) into (4.4):

$$-\frac{\mu}{\varepsilon^2}\Big[\frac{\varepsilon^2}{\mu}\Big(\frac{\partial^2 v_r^0}{\partial r^2}+\varepsilon\frac{\partial^2 v_r^1}{\partial r^2}+\varepsilon^2\frac{\partial^2 v_r^2}{\partial r^2}+\cdots\Big)+\varepsilon^2\cdot\frac{\varepsilon^2}{\mu}\Big(\frac{\partial^2 v_r^0}{\partial z^2}+\varepsilon\frac{\partial^2 v_r^1}{\partial z^2}+\varepsilon^2\frac{\partial^2 v_r^2}{\partial z^2}+\cdots\Big)$$
$$+\frac{1}{r}\cdot\frac{\varepsilon^2}{\mu}\Big(\frac{\partial v_r^0}{\partial r}+\varepsilon\frac{\partial v_r^1}{\partial r}+\varepsilon^2\frac{\partial v_r^2}{\partial r}+\cdots\Big)-\frac{1}{r^2}\cdot\frac{\varepsilon^2}{\mu}\Big(v_r^0+\varepsilon v_r^1+\varepsilon^2 v_r^2+\cdots\Big)\Big]$$
$$+\frac{1}{\varepsilon}\Big(\frac{\partial p^0}{\partial r}+\varepsilon\frac{\partial p^1}{\partial r}+\varepsilon^2\frac{\partial p^2}{\partial r}+\cdots\Big)=0.$$

we collect the terms of the same order starting from the lowest order:

$$\varepsilon^{-1}\ :\ \frac{\partial p^0}{\partial r}=0, \tag{5.1}$$

$$\varepsilon^{0}\ :\ -\Big(\frac{\partial^2 v_r^0}{\partial r^2}+\frac{1}{r}\frac{\partial v_r^0}{\partial r}-\frac{1}{r^2}v_r^0\Big)+\frac{\partial p^1}{\partial r}=0, \tag{5.2}$$

$$\varepsilon^{1}\ :\ -\Big(\frac{\partial^2 v_r^1}{\partial r^2}+\frac{1}{r}\frac{\partial v_r^1}{\partial r}-\frac{1}{r^2}v_r^1\Big)+\frac{\partial p^2}{\partial r}=0, \tag{5.3}$$

$$\varepsilon^{2}\ :\ -\Big(\frac{\partial^2 v_r^2}{\partial r^2}+\frac{1}{r}\frac{\partial v_r^2}{\partial r}+\frac{\partial^2 v_r^0}{\partial z^2}-\frac{1}{r^2}v_r^2\Big)+\frac{\partial p^3}{\partial r}=0. \tag{5.4}$$

From (5.1), we get

$$p^0 = p^0(z,t). \tag{5.5}$$

- Insert (4.27) and (4.28) into (4.5)

$$-\frac{\mu}{\varepsilon^2}\Big[\frac{\varepsilon^2}{\mu}\Big(\frac{\partial^2 v_z^0}{\partial r^2} + \varepsilon\frac{\partial^2 v_z^1}{\partial r^2} + \varepsilon^2\frac{\partial^2 v_z^2}{\partial r^2} + \cdots\Big) + \varepsilon^2 \cdot \frac{\varepsilon^2}{\mu}\Big(\frac{\partial^2 v_z^0}{\partial z^2} + \varepsilon\frac{\partial^2 v_z^1}{\partial z^2} + \varepsilon^2\frac{\partial^2 v_z^2}{\partial z^2} + \cdots\Big)$$
$$+\frac{1}{r}\cdot\frac{\varepsilon^2}{\mu}\Big(\frac{\partial v_z^0}{\partial r} + \varepsilon\frac{\partial v_z^1}{\partial r} + \varepsilon^2\frac{\partial v_z^2}{\partial r} + \cdots\Big)\Big] + \Big(\frac{\partial p^0}{\partial z} + \varepsilon\frac{\partial p^1}{\partial z} + \varepsilon^2\frac{\partial p^2}{\partial z} + \cdots\Big) = 0.$$

we collect again the terms of the same order starting from the lowest order:

$$\varepsilon^0 \;:\; -\Big(\frac{\partial^2 v_z^0}{\partial r^2} + \frac{1}{r}\frac{\partial v_z^0}{\partial r}\Big) + \frac{\partial p^0}{\partial z} = 0, \tag{5.6}$$

$$\varepsilon^1 \;:\; -\Big(\frac{\partial^2 v_z^1}{\partial r^2} + \frac{1}{r}\frac{\partial v_z^1}{\partial r}\Big) + \frac{\partial p^1}{\partial z} = 0, \tag{5.7}$$

$$\varepsilon^2 \;:\; -\Big(\frac{\partial^2 v_z^2}{\partial r^2} + \frac{1}{r}\frac{\partial v_z^2}{\partial r} + \frac{\partial^2 v_z^0}{\partial z^2}\Big) + \frac{\partial p^2}{\partial z} = 0. \tag{5.8}$$

- Insert (4.27) into (4.6):

$$\frac{\varepsilon^2}{\mu}\Big(\frac{\partial v_r^0}{\partial r} + \varepsilon\frac{\partial v_r^1}{\partial r} + \varepsilon^2\frac{\partial v_r^2}{\partial r} + \cdots\Big) + \frac{\varepsilon^2}{\mu}\cdot\varepsilon\Big(\frac{\partial v_z^0}{\partial z} + \varepsilon\frac{\partial v_z^1}{\partial z} + \varepsilon^2\frac{\partial v_z^2}{\partial z} + \cdots\Big)$$
$$+\frac{\varepsilon^2}{\mu}\cdot\frac{1}{r}\Big(v_r^0 + \varepsilon v_r^1 + \varepsilon^2 v_r^2 + \cdots\Big) = 0$$

we collect again the terms of the same order:

$$\varepsilon^0 \;:\; \frac{\partial v_r^0}{\partial r} + \frac{1}{r}v_r^0 = 0, \tag{5.9}$$

$$\varepsilon^1 \;:\; \frac{\partial v_r^1}{\partial r} + \frac{1}{r}v_r^1 + \frac{\partial v_z^0}{\partial z} = 0, \tag{5.10}$$

$$\varepsilon^2 \;:\; \frac{\partial v_r^2}{\partial r} + \frac{1}{r}v_r^2 + \frac{\partial v_z^1}{\partial z} = 0. \tag{5.11}$$

Using (5.9), we conclude that $\frac{1}{r}\frac{\partial(rv_r^0)}{\partial r} = 0$. This yields $rv_r^0 = constant$ as $r \to 0$. Physically this says that only $constant \equiv 0$ has meaning, and we get

$$v_r^0 = 0 \tag{5.12}$$

Notice that (5.12) indicates that in this coupled fluid-structure problem for creeping flow, the radial component of the velocity is by one order of magnitude smaller than the axial component.

By inserting (5.12) into (5.2), we get $\frac{\partial p^1}{\partial r} = 0$, which says that p^1 is independent of r, i.e.

$$p^1 = p^1(z,t), \tag{5.13}$$

• Insert (4.27) and (4.29) to the lateral boundary conditions, we can get:

$$\frac{\varepsilon^2}{\mu}(v_r^0 + \varepsilon v_r^1 + \varepsilon^2 v_r^2 + \cdots) = \frac{\varepsilon^2}{\mu}\varepsilon\frac{\partial}{\partial t}(\eta^0 + \varepsilon\eta^1 + \varepsilon^2\eta^2 + \cdots) \tag{5.14}$$

$$\frac{\varepsilon^2}{\mu}(v_z^0 + \varepsilon v_z^1 + \varepsilon^2 v_z^2 + \cdots) = \frac{\varepsilon^2}{\mu}\frac{\partial}{\partial t}(s^0 + \varepsilon s^1 + \varepsilon^2 s^2 + \cdots) \tag{5.15}$$

In order to simplify the notation, we introduce

$$p = p^0 + \varepsilon p^1, s = s^0 + \varepsilon s^1, \eta = \eta^0 + \varepsilon\eta^1, v_r = v_r^1 + \varepsilon v_r^2, v_z = v_z^0 + \varepsilon v_z^1$$

(5.5) and (5.13) implies

$$p = p(z,t) \tag{5.16}$$

(5.14) and (5.15) imply that

$$v_r(z,R,t) = \frac{\partial\eta}{\partial t}(z,t) \tag{5.17}$$

$$v_z(z,R,t) = \frac{\partial s}{\partial t}(z,t) \tag{5.18}$$

(5.10)+$\varepsilon\times$(5.11), using the notation $v_r = v_r^1 + \varepsilon v_r^2$, imply:

$$\frac{\partial}{\partial r}(rv_r) + \frac{\partial}{\partial z}(rv_z) = 0 \tag{5.19}$$

(5.6)+$\varepsilon\times$(5.7),using the notation $p = p^0 + \varepsilon p^1$ and $v_z = v_z^0 + \varepsilon v_z^1$, imply:

$$r\frac{\partial p}{\partial z} = \frac{\partial}{\partial r}(r\frac{\partial v_z}{\partial r}) \tag{5.20}$$

Equations (5.2) (5.19) and (5.20) are the standard asymptotic equations obtained from the flow equations before any boundary conditions are taken into account.

• We insert (4.28), (4.29) and (4.30) into (2.2) and (4.11), and obtain :

$$F_r = -\frac{h(\varepsilon)E(\varepsilon)}{(1-\sigma^2)\varepsilon}\left[\frac{\sigma}{R}\frac{\partial}{\partial z}(s^0 + \varepsilon s^1 + \cdots) + \frac{1}{R^2}(\eta^0 + \varepsilon\eta^1 + \cdots)\right]$$
$$+h(\varepsilon)G(\varepsilon)k(\varepsilon)\varepsilon\frac{\partial^2}{\partial z^2}(\eta^0 + \varepsilon\eta^1 + \cdots) - \rho_\omega h(\varepsilon)\frac{\varepsilon^4}{\mu^2}\varepsilon\frac{\partial^2}{\partial t^2}(\eta^0 + \varepsilon\eta^1 + \cdots)$$

$$F_r = -(p^0 + \varepsilon p^1 + \cdots) + 2\mu\frac{1}{\varepsilon}\frac{\partial}{\partial r}\left(\frac{\varepsilon^2}{\mu}(\varepsilon v_r^1 + \varepsilon^2 v_r^2 + \cdots)\right)$$

By using the assumptions, (2.5) and (2.6) as $\varepsilon \to 0$ with the notation $p = p^0 + \varepsilon p^1, s = s^0 + \varepsilon s^1, \eta = \eta^0 + \varepsilon \eta^1$, we get:

$$p = \frac{E_0}{R(1-\sigma^2)}\Big[\sigma \frac{\partial s}{\partial z} + \frac{\eta}{R}\Big] - G_0 \frac{\partial^2 \eta}{\partial z^2} + \mathcal{O}(\varepsilon^2) \tag{5.21}$$

• We next insert (4.28), (4.29) and (4.30) into (2.3) to obtain:

$$\begin{aligned} F_z = \frac{h(\varepsilon)E(\varepsilon)}{1-\sigma^2}\Big[\frac{\partial^2}{\partial z^2}(s^0 + \varepsilon s^1 + \cdots) + \frac{\sigma}{\varepsilon R}\frac{\partial}{\partial z}(\varepsilon(\eta^0 + \varepsilon \eta^1 + \cdots))\Big] \\ -\rho_\omega h(\varepsilon)\frac{\varepsilon^4}{\mu^2}\frac{\partial^2}{\partial t^2}(s^0 + \varepsilon s^1 + \cdots) \end{aligned}$$

$$F_z = \mu\bigg(\frac{\partial}{\partial z}\Big(\frac{\varepsilon^2}{\mu}(\varepsilon v_r^1 + \varepsilon^2 v_r^2 + \cdots)\Big) + \frac{1}{\varepsilon}\frac{\partial}{\partial r}\Big(\frac{\varepsilon^2}{\mu^2}(v_z^0 + \varepsilon v_z^1 + \cdots)\Big)\bigg)$$

Using the assumptions (2.4), (2.5) and (2.6), we get:

$$\frac{\partial}{\partial r}\big(v_z^0 + \varepsilon v_z^1 + \cdots\big) = E_0 \frac{1}{1-\sigma^2}\Big(\frac{\partial^2}{\partial z^2}(s^0 + \varepsilon s^1 + \cdots) + \frac{\sigma}{R}\frac{\partial}{\partial z}(\eta^0 + \varepsilon \eta^1 + \cdots)\Big)$$

we get:

$$\frac{\partial v_z}{\partial r}|_{r=R} = \frac{E_0}{1-\sigma^2}\frac{\partial}{\partial z}\Big(\frac{\partial s}{\partial z} + \frac{\sigma}{R}\eta\Big) + \mathcal{O}(\varepsilon^2) \tag{5.22}$$

We now focus on (5.21) and (5.22). Our goal is to obtain a PDE for the pressure only. So our task is to eliminate η, v_z and $\frac{\partial s}{\partial z}$ from (5.21) and (5.22).

• For the equation (5.20), first integrate from 0 to r, then integrate again from r to R and use the continuity at the boundary (5.18), i.e. $v_z(z,R,t) = \frac{\partial s}{\partial t}(z,t)$ to get

$$v_z(r,z,t) = \frac{R^2 - r^2}{4}\frac{\partial p}{\partial z}(z,t) + \frac{\partial s}{\partial t}(z,t). \tag{5.23}$$

• From (5.23), we get

$$\frac{\partial v_z}{\partial r} = \frac{\partial p}{\partial z}\frac{r}{2} \Rightarrow \frac{\partial v_z}{\partial r}|_{r=R} = \frac{\partial p}{\partial z}\frac{R}{2} \tag{5.24}$$

Combining (5.24) with (5.22), we get

$$\frac{R}{2}\frac{\partial p}{\partial z} = \frac{E_0}{1-\sigma^2}\frac{\partial}{\partial z}\Big(\frac{\partial s}{\partial z} + \frac{\sigma}{R}\eta\Big) \tag{5.25}$$

An integration on both sides of (5.25) for z from 0 to z, yields

$$\frac{R}{2}p(z,t) = \frac{E_0}{1-\sigma^2}\Big(\frac{\partial s}{\partial z} + \frac{\sigma}{R}\eta\Big) \tag{5.26}$$

- An integration of (5.19) from 0 to R in r, yields

$$\frac{\partial\eta}{\partial t} - \frac{R^3}{16}\frac{\partial^2 p}{\partial z^2} + \frac{R}{2}\frac{\partial^2 s}{\partial z\partial t} = 0 \tag{5.27}$$

and an integration with respect to t, gives

$$\frac{\partial s}{\partial z} = \frac{R^2}{8}\frac{\partial^2}{\partial z^2}\Big(\int_0^t pdt\Big) - \frac{2\eta}{R} \tag{5.28}$$

We can now insert (5.28) to (5.26), which gives

$$\frac{\eta}{R} = \frac{R(1-\sigma^2)}{E_0(2-\sigma)}\left(\frac{RE_0}{8(1-\sigma^2)}\frac{\partial^2}{\partial z^2}\Big(\int_0^t pdt\Big) - \frac{p}{2}\right) \tag{5.29}$$

By inserting the expression for $\frac{\eta}{R}$ into (5.28), we obtain

$$\frac{\partial s}{\partial z} = \left(\frac{R^2}{8} - \frac{R^2}{4(2-\sigma)}\right)\frac{\partial^2}{\partial z^2}\Big(\int_0^t pdt\Big) - \frac{pR(1-\sigma^2)}{(2-\sigma)E_0}. \tag{5.30}$$

- Further by using (5.21), we get

$$\frac{\partial p}{\partial z}(z,t) = \frac{E_0}{R(1-\sigma^2)}\frac{\partial}{\partial z}\left[\sigma\frac{\partial s}{\partial z} + \frac{\eta}{R}\right] - G_0\frac{\partial}{\partial z}\Big(\frac{\partial^2\eta}{\partial z^2}\Big) \tag{5.31}$$

insert (5.31) into (5.25),implies

$$\frac{2}{R}\frac{E_0}{1-\sigma^2}\Big(\frac{\partial s}{\partial z} + \frac{\sigma}{R}\eta\Big) = \frac{E_0}{R(1-\sigma^2)}\frac{\partial}{\partial z}\left[\sigma\frac{\partial s}{\partial z} + \frac{\eta}{R}\right] - G_0\frac{\partial}{\partial z}\Big(\frac{\partial^2\eta}{\partial z^2}\Big)$$

$$\frac{E_0(2-\sigma)}{R(1-\sigma^2)}\frac{\partial^2 s}{\partial z^2} + \frac{E_0(2\sigma-1)}{R(1-\sigma^2)}\frac{\partial}{\partial z}\Big(\frac{\eta}{R}\Big) = -G_0\frac{\partial}{\partial z}\Big(\frac{\partial^2\eta}{\partial z^2}\Big)$$

integrating with respect to z from 0 to z, we obtain

$$\frac{E_0(2-\sigma)}{R(1-\sigma^2)}\frac{\partial s}{\partial z} + \frac{E_0(2\sigma-1)}{R(1-\sigma^2)}\frac{\eta}{R} = -G_0\frac{\partial^2\eta}{\partial z^2} \tag{5.32}$$

Finally, we insert (5.29) and (5.30) into (5.32) to arrive at

$$\frac{\partial}{\partial t}\left(\Big(\frac{5}{2} - 2\sigma\Big)p - (1-\sigma^2)\frac{G_0R^2}{2E_0}\frac{\partial^2 p}{\partial z^2}\right) = \frac{\partial^2}{\partial z^2}\left(\frac{E_0Rp}{8} - \frac{G_0R^3}{8}\frac{\partial^2 p}{\partial z^2}\right) \tag{5.33}$$

Depending on the problem, the coefficients containing shear modulus G_0 may or may not be negligible. In the following two subsection, we summarize the initial-boundary-value problems corresponding to the two cases. But before do this, first we show how to deduce the reduced equation by using the **Asymptotic expansion II**, only differences are the Stokes equaitons, which will be showed next step.

5.1.2 Using the *Asymptotic Expansions II* to get the reduced equations of Stokes equations

First recall **Asymptotic expansion II**

$$v(\varepsilon)_r(z,r,t) = \frac{\varepsilon^3}{\mu}\sum_{i\geq 0}\varepsilon^i v_r^i(z,r,t), \tag{5.34}$$

$$v(\varepsilon)_z(z,r,t) = \frac{\varepsilon^2}{\mu}\sum_{i\geq 0}\varepsilon^i v_z^i(z,r,t), \tag{5.35}$$

$$p(\varepsilon)(z,r,t) = \sum_{i\geq 0}\varepsilon^i p^i(z,r,t). \tag{5.36}$$

- Insert (5.34) and (5.36) to (4.4)

$$-\frac{\mu}{\varepsilon^2}\Big[\frac{\varepsilon^3}{\mu}\Big(\frac{\partial^2 v_r^0}{\partial r^2}+\varepsilon\frac{\partial^2 v_r^1}{\partial r^2}+\varepsilon^2\frac{\partial^2 v_r^2}{\partial r^2}+\cdots\Big)+\varepsilon^2\cdot\frac{\varepsilon^3}{\mu}\Big(\frac{\partial^2 v_r^0}{\partial z^2}+\varepsilon\frac{\partial^2 v_r^1}{\partial z^2}+\varepsilon^2\frac{\partial^2 v_r^2}{\partial z^2}+\cdots\Big)$$
$$+\frac{1}{r}\cdot\frac{\varepsilon^3}{\mu}\Big(\frac{\partial v_r^0}{\partial r}+\varepsilon\frac{\partial v_r^1}{\partial r}+\varepsilon^2\frac{\partial v_r^2}{\partial r}+\cdots\Big)-\frac{1}{r^2}\cdot\frac{\varepsilon^3}{\mu}\Big(v_r^0+\varepsilon v_r^1+\varepsilon^2 v_r^2+\cdots\Big)\Big]$$
$$+\frac{1}{\varepsilon}\Big(\frac{\partial p^0}{\partial r}+\varepsilon\frac{\partial p^1}{\partial r}+\varepsilon^2\frac{\partial p^2}{\partial r}+\cdots\Big)=0$$

Collecting the terms of the same order starting from the lowest order results in:

$$\varepsilon^{-1} \; : \; \frac{\partial p^0}{\partial r}=0 \tag{5.37}$$

$$\varepsilon^{0} \; : \; \frac{\partial p^1}{\partial r}=0 \tag{5.38}$$

$$\varepsilon^{1} \; : \; -\Big(\frac{\partial^2 v_r^0}{\partial r^2}+\frac{1}{r}\frac{\partial v_r^0}{\partial r}-\frac{1}{r^2}v_r^0\Big)+\frac{\partial p^2}{\partial r}=0 \tag{5.39}$$

From (5.37) and (5.38) , we conclude

$$p^0=p^0(z,t)\ ,\ p^1=p^1(z,t) \tag{5.40}$$

- Now insert (5.35) and (5.36) into (4.5)

$$-\frac{\mu}{\varepsilon^2}\Big[\frac{\varepsilon^2}{\mu}\Big(\frac{\partial^2 v_z^0}{\partial r^2}+\varepsilon\frac{\partial^2 v_z^1}{\partial r^2}+\varepsilon^2\frac{\partial^2 v_z^2}{\partial r^2}+\cdots\Big)+\varepsilon^2\cdot\frac{\varepsilon^2}{\mu}\Big(\frac{\partial^2 v_z^0}{\partial z^2}+\varepsilon\frac{\partial^2 v_z^1}{\partial z^2}+\varepsilon^2\frac{\partial^2 v_z^2}{\partial z^2}+\cdots\Big)$$
$$+\frac{1}{r}\cdot\frac{\varepsilon^2}{\mu}\Big(\frac{\partial v_z^0}{\partial r}+\varepsilon\frac{\partial v_z^1}{\partial r}+\varepsilon^2\frac{\partial v_z^2}{\partial r}+\cdots\Big)\Big]+\Big(\frac{\partial p^0}{\partial z}+\varepsilon\frac{\partial p^1}{\partial z}+\varepsilon^2\frac{\partial p^2}{\partial z}+\cdots\Big)=0$$

Collecting the terms of the same order starting from the lowest order again yields:

$$\varepsilon^0 \quad : \quad -\Big(\frac{\partial^2 v_z^0}{\partial r^2} + \frac{1}{r}\frac{\partial v_z^0}{\partial r}\Big) + \frac{\partial p^0}{\partial z} = 0, \tag{5.41}$$

$$\varepsilon^1 \quad : \quad -\Big(\frac{\partial^2 v_z^1}{\partial r^2} + \frac{1}{r}\frac{\partial v_z^1}{\partial r}\Big) + \frac{\partial p^1}{\partial z} = 0, \tag{5.42}$$

$$\varepsilon^2 \quad : \quad -\Big(\frac{\partial^2 v_z^2}{\partial r^2} + \frac{1}{r}\frac{\partial v_z^2}{\partial r} + \frac{\partial^2 v_z^0}{\partial z^2}\Big) + \frac{\partial p^2}{\partial z} = 0. \tag{5.43}$$

- Next insert (5.34) and (5.35) into (4.6)

$$\frac{\varepsilon^3}{\mu}\Big(\frac{\partial v_r^0}{\partial r} + \varepsilon\frac{\partial v_r^1}{\partial r} + \varepsilon^2\frac{\partial v_r^2}{\partial r} + \cdots\Big) + \frac{\varepsilon^2}{\mu}\cdot\varepsilon\Big(\frac{\partial v_z^0}{\partial z} + \varepsilon\frac{\partial v_z^1}{\partial z} + \varepsilon^2\frac{\partial v_z^2}{\partial z} + \cdots\Big)$$
$$+\frac{\varepsilon^3}{\mu}\cdot\frac{1}{r}\Big(v_r^0 + \varepsilon v_r^1 + \varepsilon^2 v_r^2 + \cdots\Big) = 0$$

Collecting again the terms of the same order gives:

$$\varepsilon^3 \quad : \quad \frac{\partial v_r^0}{\partial r} + \frac{1}{r}v_r^0 + \frac{\partial v_z^0}{\partial z} = 0 \tag{5.44}$$

$$\varepsilon^4 \quad : \quad \frac{\partial v_r^1}{\partial r} + \frac{1}{r}v_r^1 + \frac{\partial v_z^1}{\partial z} = 0 \tag{5.45}$$

$$\varepsilon^5 \quad : \quad \frac{\partial v_r^2}{\partial r} + \frac{1}{r}v_r^2 + \frac{\partial v_z^2}{\partial z} = 0 \tag{5.46}$$

We use the following notation for v_r and the others are as same as before.

$$v_r = v_r^0 + \varepsilon v_r^1$$

From (5.37) and (5.38), we deduce

$$p = p(z,t)$$

From (5.41) and (5.42), we deduce

$$r\frac{\partial p}{\partial z} = \frac{\partial}{\partial r}(r\frac{\partial v_z}{\partial r})$$

From (5.44) and (5.45), we deduce

$$\frac{\partial}{\partial r}(rv_r) + \frac{\partial}{\partial z}(rv_z) = 0$$

And we found these are exactly the same as using the **Asymptotic expansion I**. The remaining part to derive the pressure equation is analogous.

5.2 The reduced problem $\mathbf{P}_1$ when shear modulus G_0 is 0 or negligible

In this subsection we study the case when the coefficients containing shear modulus G_0 is zero or negligible. After taking into account the obvious regularity of p^ε with respect to z, we see that $p = A(t)$ for $z = L$. The reduced initial-boundary-value problem for the effective pressure reads

$$\begin{cases} \left(\frac{5}{2} - 2\sigma\right)\frac{\partial p}{\partial t} = \frac{E_0 R}{8}\frac{\partial^2 p}{\partial z^2} & in\ (0, L) \times (0, T) \\ p(0, t) = 0,\ p(L, t) = A(t) & in\ (0, T) \\ p(z, 0) = 0 & in\ (0, L) \end{cases} \tag{5.47}$$

and the relationship between $\frac{\partial s}{\partial z}$ and η becomes

$$\frac{\partial s}{\partial z} = \frac{1 - 2\sigma}{2 - \sigma}\frac{\eta}{R} \tag{5.48}$$

We insert (5.48) into (5.21) and get:

$$\eta = \frac{R^2(2 - \sigma)}{2E_0}p, \qquad \frac{\partial s}{\partial z} = \frac{R(1 - 2\sigma)}{2E_0}p$$

Then we get the radius of the tube is $r = R + \eta$. We will plot the pressure $p = P_0 + A(t)$ where P_0 is the reference pressure. It is well-know that equations (5.47) has a unique smooth solution p.

For incompressible materials, $\sigma = 1/2$, in which case (5.48) implies $s = 0$ and from (5.25), with $\sigma = 1/2$ we have

$$p = \frac{4E_0}{3R^2}\eta$$

This is the Law of Laplace or the independent ring model, found in[19]

In general for the negligible shear modulus, the pressure is directly related to the radial displacement via

$$p = \frac{2E_0}{R^2(2 - \sigma)}\eta$$

We see that , for general σ, the diffusion equation for the effective pressure can be easily written in terms of the radial desplacement.The resulting equation is parabolic, reflecting the fact that acceleration terms in the fluid equation have been ignored. More precisely, if the acceleration terms were present, the resulting equaitons would include the second derivative of η with respect to time and give a hyperbolic problem. Hyperbolic problems are typically obtained when the reduced Navier-Stokes equations are coupled with the independent ring model, see[4;19].

Here we will show the numerical simulation and the pressure drop A(t) prescribed on the right boundary $z = l$, given by $A(t) = 950\sin(2\pi t)Pa$. Figure 5.1 shows that pressure distribution using $CG1$ to simulate. Figure 5.2 and Figure 5.3 show how pressure and displacements change at different time.

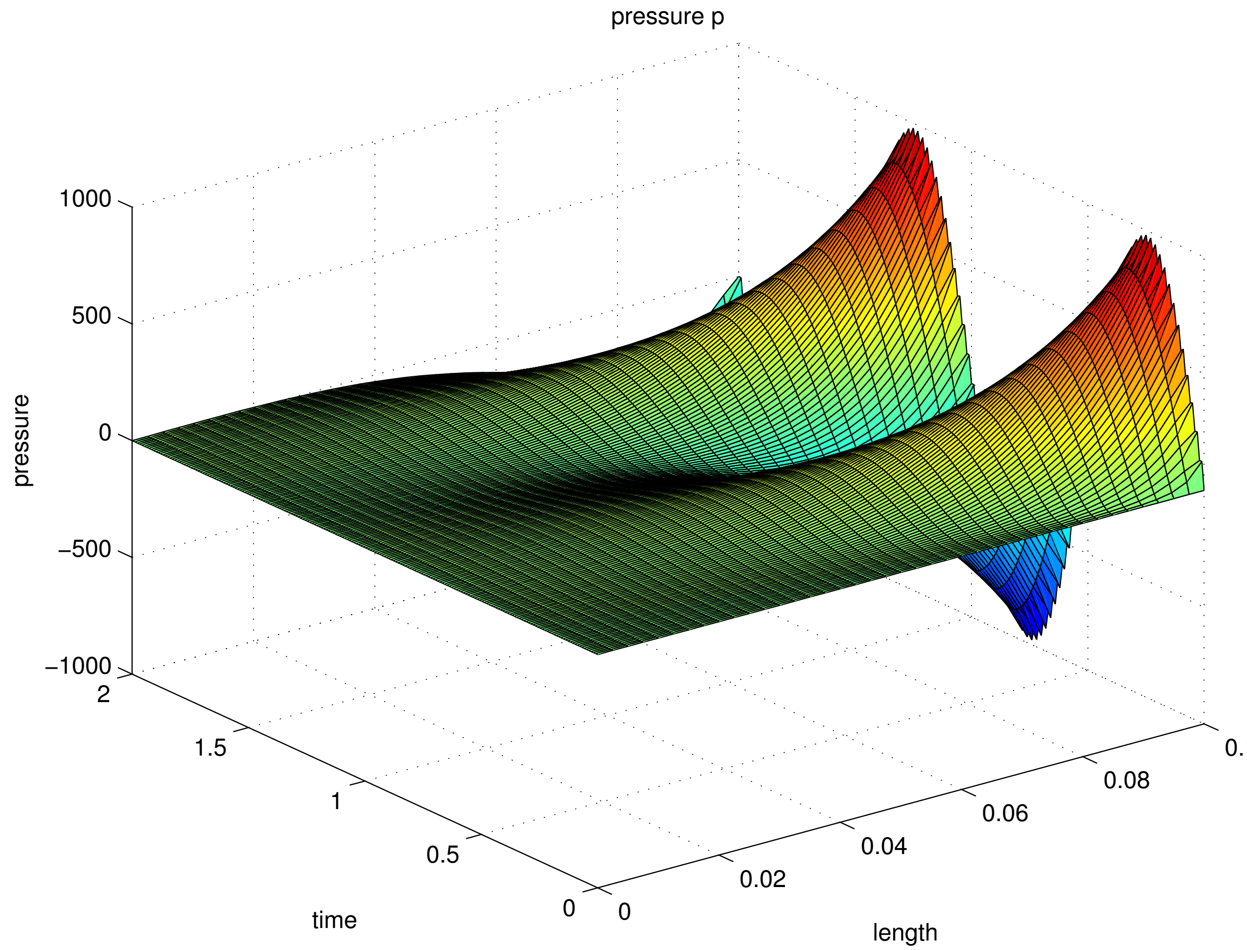


Figure 5.1: pressure distribution of $\mathbf{P}_1$.

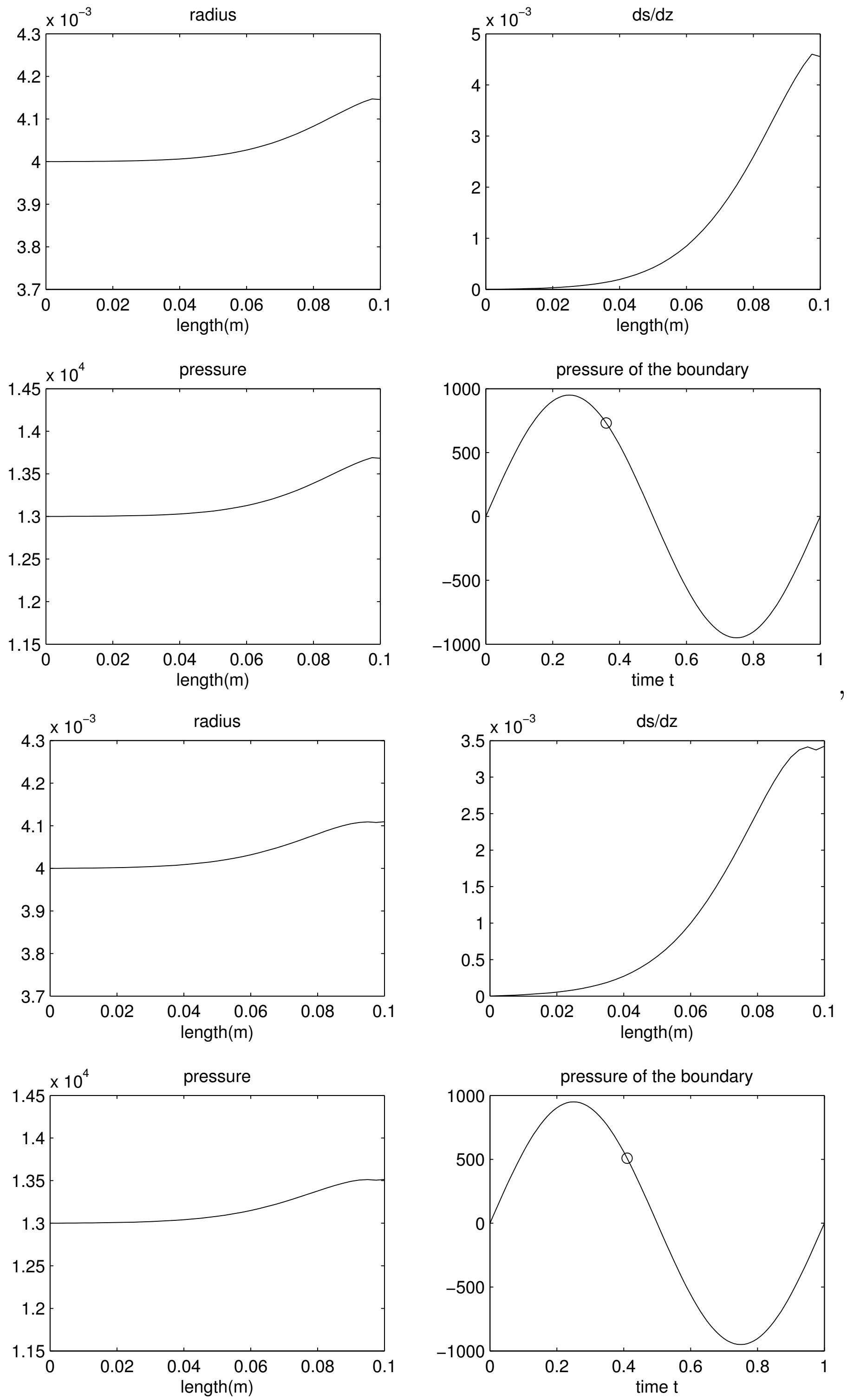


Figure 5.2: Effective displacements for $\mathbf{P_1}$ with the given pressure drop A(t). The reference pressure is 13000 Pa, above figure at t=0.35, down figure at t=0.42.

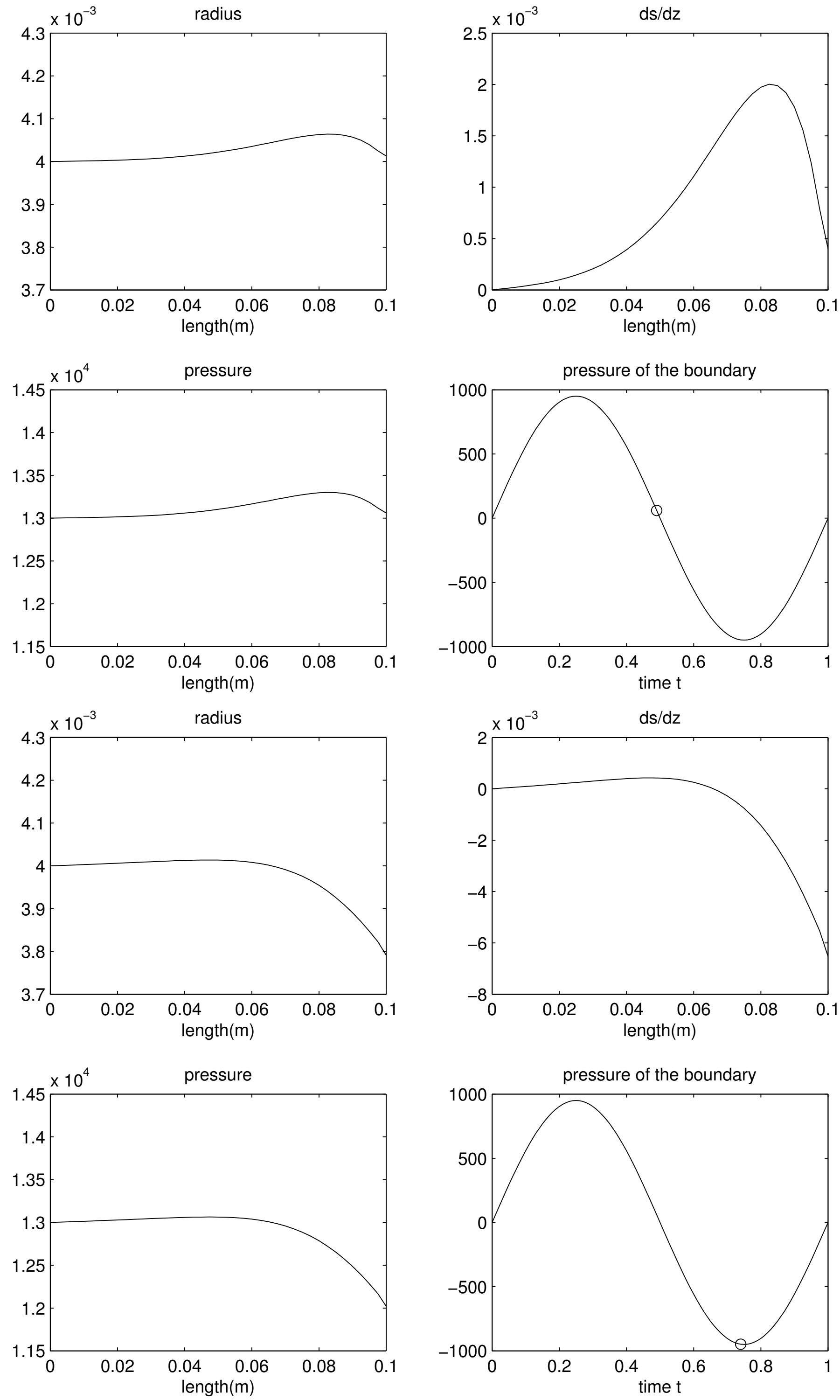


Figure 5.3: Effective displacements for $\mathbf{P_1}$ at another time, t=0.5 and t=0.75

5.3 The reduced problem P_2 for nonnegligible shear modulus

In this case when the shear modulus coefficients are not small, we need more boundary conditions for(5.33). Furthermore,for $G_0 > 0$ the boundary conditions for the radial displacement are preserved in the limit. By using (5.29) we get the boundary conditions for $\partial_{zz}p$ at $z = 0, L$.

$$\begin{cases} \frac{\partial}{\partial t}\Big(\big(\frac{5}{2} - 2\sigma\big)p - (1-\sigma^2)\frac{G_0 R^2}{2E_0}\frac{\partial^2 p}{\partial z^2}\Big) = \frac{\partial^2}{\partial z^2}\Big(\frac{E_0 R}{8}p - \frac{G_0 R^3}{8}\frac{\partial^2 p}{\partial z^2}\Big) \\ p(0,t) = 0, \quad p(L,t) = A(t), \quad t \in (0,T) \\ \frac{\partial^2 p}{\partial z^2}(0,t) = 0, \quad \frac{E_0 R}{8(1-\sigma^2)}\frac{\partial^2 p}{\partial z^2}(L,t) = \frac{1}{2}\frac{dA}{dt}, \quad t \in (0,T) \\ p(z,0) = 0, \quad z \in (0,L) \end{cases} \tag{5.49}$$

Remark: If $A \in C_0^\infty(0,+\infty)$,then $p \in C^\infty([0,L]\times[0,T])$.

5.4 The reduced problem P_3 in the pressure-velocity form

It is useful to cast the above reduced problem in terms of the leading order velocity and pressure. In next section, we will show that the solution of the original problem converges to the solution of the reduced problem written in terms of (v_z, p, η, s)

Find (v_z, p, η, s) such that the following equations describing conservation of mass and momentum hold :

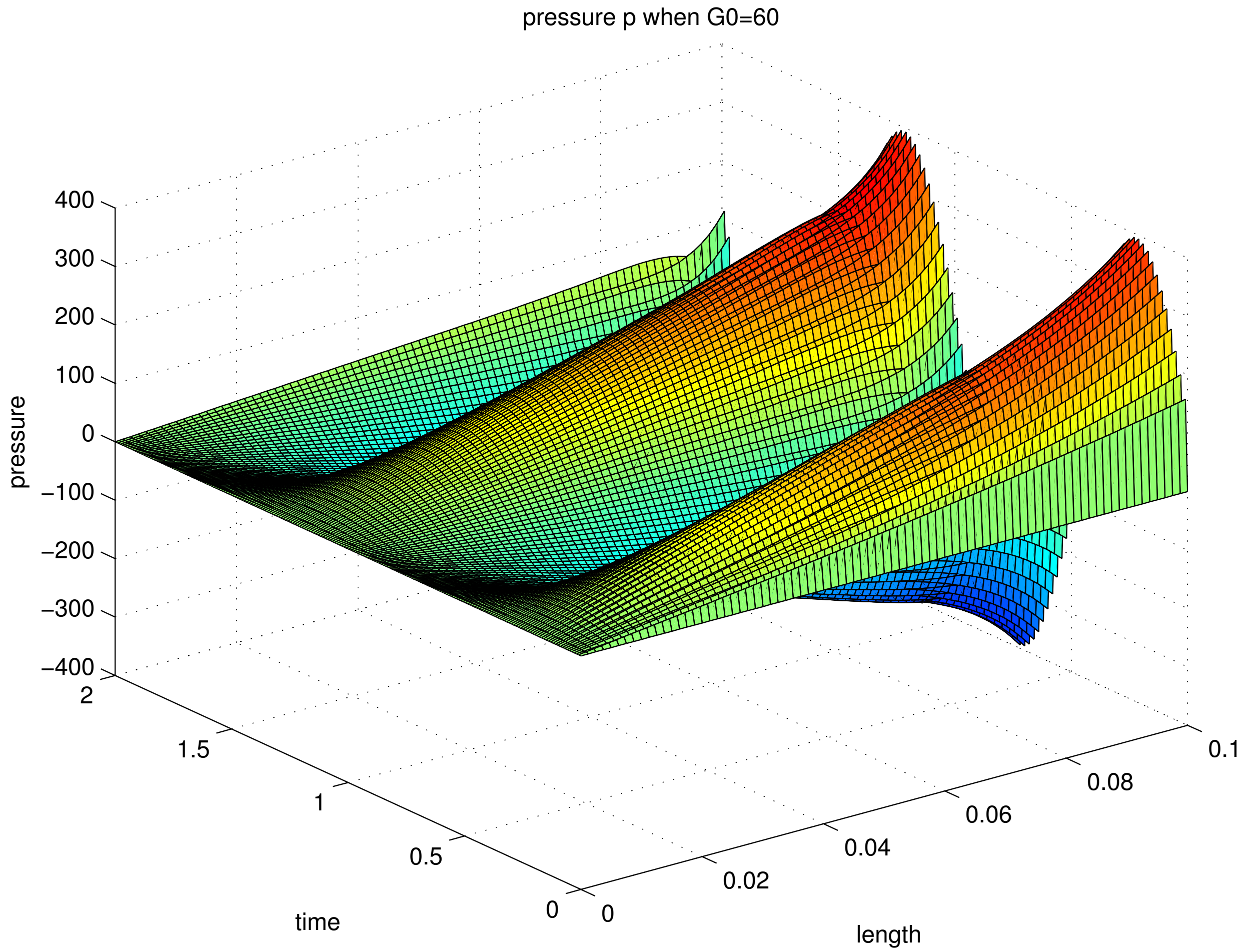
$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial z}\left(\frac{1}{R}\int_0^R v_z(r,z,t)\,r\,dr\right) = 0$$
$$r\frac{\partial p}{\partial z} = \frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)$$

the lateral boundary conditions are

$$v_z(z,R,t) = \frac{\partial s}{\partial t}(z,t),$$
$$p(z,t) = \frac{E_0}{R(1-\sigma^2)}\left(\sigma\frac{\partial s}{\partial z} + \frac{\eta}{R}\right) - G_0\frac{\partial^2 \eta}{\partial z^2}$$
$$\left.\frac{\partial v_z}{\partial r}\right|_{r=R} = \frac{E_0}{1-\sigma^2}\frac{\partial}{\partial z}\left(\frac{\partial s}{\partial z} + \frac{\sigma}{R}\eta\right)$$

and the inlet and outlet boundary data and the initial data are given by

$$\eta(0,t) = p(0,t) = \frac{\partial s}{\partial z}(0,t) = 0$$
$$\eta(L,t) = s(L,t) = 0, \qquad p(L,t) = A(t)$$
$$\eta(z,0) = s(z,0) = 0$$



,

Figure 5.4: pressure distribution of $\mathbf{P}_2$ when $G_0 = 60$. Note: here we use Quadratic basis (CG2 FEM and a *Crank-Nicolson* system) to simulate since we have fourth order derivative. Remarks : The coefficents of the fourth order term $(1-\sigma^2)\frac{G_0 R^2}{2E_0}$ and $\frac{G_0 R^3}{8}$ are very small and thus the effect of these two terms are not so visible. However, (5.49) also has boundary value for $\frac{\partial^2 p}{\partial z^2}$ specified which will have an impact on the solution. Later we will see how it changes when G_0 is changed.

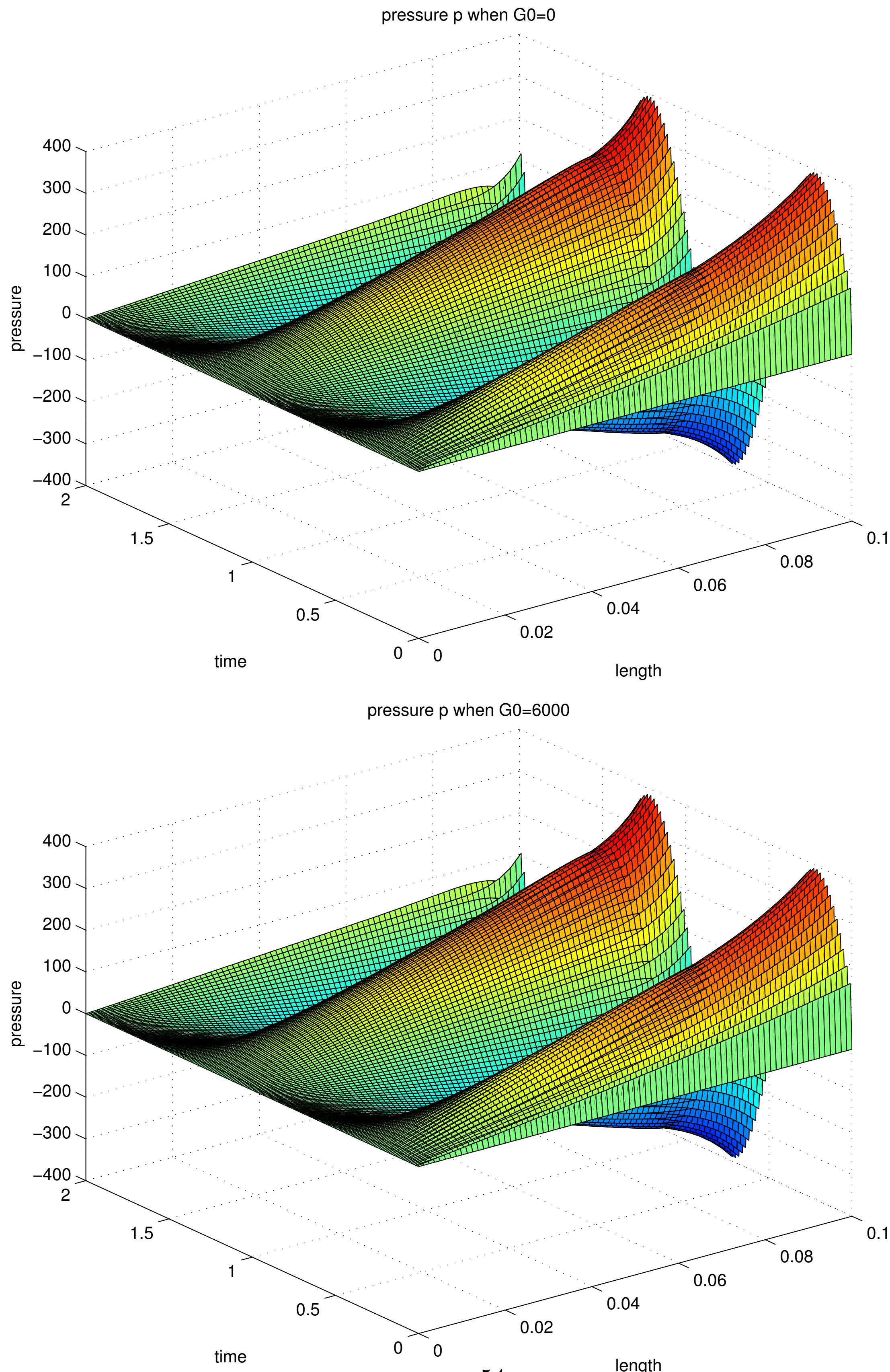


Figure 5.5: pressure distribution of $\mathbf{P}_2$. Above: $G_0 = 0$; Down: $G_0 = 6000$

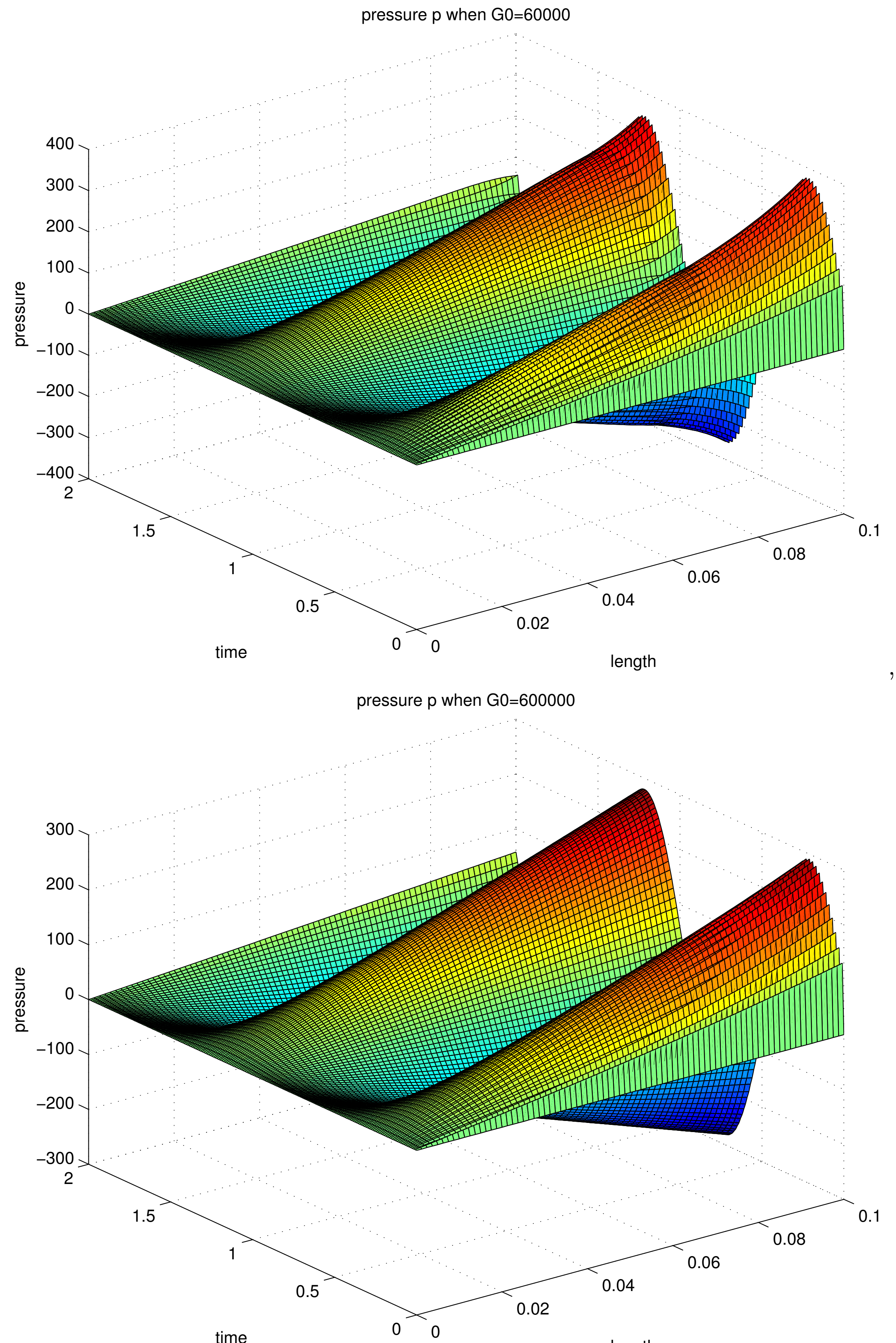


Figure 5.6: pressure distribution of $\mathbf{P}_2$. Above: $G_0 = 60000$; Down: $G_0 = 600000$

Chapter 6

Convergence Theorem

In this section we study the rescaled problem $P(\varepsilon)$ in the limit situation when $\varepsilon \to 0$. And in this section, we will deduce the form of the limit problem namely we call it $P(\varepsilon \to 0)$.

The following we will verify that the rescaled problem converges to the reduced problem. We will depend on the a priori estimates for the variables. But first we would like recall the a priori estimates for $(v(\varepsilon)_r, v(\varepsilon)_z, p(\varepsilon), \eta^\varepsilon, s^\varepsilon)$ which was used to to determine the asymptotic expansions.

$$\begin{aligned}
\|v(\varepsilon)_r\|_{L^2(\Omega\times(0,T))} &\leqslant C\frac{\varepsilon^3}{\mu}\|A\|_v \\
\|v(\varepsilon)_z\|_{L^2(\Omega\times(0,T))} &\leqslant C\frac{\varepsilon^2}{\mu}\|A\|_v \\
\|v(\varepsilon)\|_{L^2(\Omega\times(0,T))} &\leqslant C\frac{\varepsilon^2}{\mu}\|A\|_v \\
\|p(\varepsilon)\|_{L^2(\Omega\times(0,T))} &\leqslant C\|A\|_v \\
\|\eta^\varepsilon(t)\|_{L^2(0,L)} &\leqslant C\varepsilon\|A\|_v \\
\|s^\varepsilon(t)\|_{L^2(0,L)} &\leqslant C\|A\|_v
\end{aligned}$$

1.
$$F_r = -\underbrace{\frac{h(\varepsilon)E(\varepsilon)}{(1-\sigma^2)\varepsilon}}_{1}\Big(\underbrace{\frac{\sigma}{R}\frac{\partial s^\varepsilon}{\partial z}}_{2}+\underbrace{\frac{\eta^\varepsilon}{\varepsilon R^2}}_{3}\Big)+\underbrace{h(\varepsilon)G(\varepsilon)k(\varepsilon)\frac{\partial^2\eta^\varepsilon}{\partial z^2}}_{4}-\underbrace{\rho_\omega h(\varepsilon)\frac{\varepsilon^4}{\mu^2}\frac{\partial^2\eta^\varepsilon}{\partial t^2}}_{5},$$

- As $\varepsilon \to 0$, we have **term 1** $\backsim 1 = \varepsilon^0$
- From $\|s^\varepsilon\|_{L^2(0,L)} \leqslant C\|A\|_v$, we have $\frac{\partial s^\varepsilon}{\partial z} \backsim 1 = \varepsilon^0$, so **term 2** $\backsim 1 = \varepsilon^0$
- From $\frac{1}{\varepsilon}\|\eta^\varepsilon(t)\|_{L^2(0,L)} \leqslant C\|A\|_v$, we have $\eta^\varepsilon \backsim \varepsilon$, so **term 3** $\backsim 1 = \varepsilon^0$
- Since the assumption is $lim_{\varepsilon\to 0}h(\varepsilon)G(\varepsilon)k(\varepsilon)\varepsilon = G_0$, so $h(\varepsilon)G(\varepsilon)k(\varepsilon) \backsim \frac{1}{\varepsilon}$, so we have **term 4** $\backsim 1 = \varepsilon^0$
- We have $h(\varepsilon) \backsim \varepsilon$, so **term 5** $\backsim 1 = \varepsilon^6$

- *We take care of the terms which have the same order of ε^0, then* ***term 5*** *will disappear.*

2.

$$F_r = -p(\varepsilon) + \underbrace{2\mu\frac{1}{\varepsilon}\frac{\partial v(\varepsilon)_r}{\partial r}}_{6}$$

- From $\|p(\varepsilon)\|_{L^2(\Omega\times(0,T))} \leqslant C\|A\|_v$, we have $p(\varepsilon) \backsim 1 = \varepsilon^0$
- From $\|v(\varepsilon)_r\|_{L^2(\Omega\times(0,T))} \leqslant C\frac{\varepsilon^3}{\mu}\|A\|_v$, we have $\frac{\partial v(\varepsilon)_r}{\partial r} \backsim \varepsilon^3$, so **term 6** $\backsim \varepsilon^2$
- *We take care of the terms which have the same order of ε^0, then* ***term 6*** *will disappear.*

3.

$$F_z = \underbrace{\frac{h(\varepsilon)E(\varepsilon)}{1-\sigma^2}}_{7}\Big(\underbrace{\frac{\partial^2 s^\varepsilon}{\partial z^2}}_{8} + \underbrace{\frac{\sigma}{\varepsilon R}\frac{\partial \eta^\varepsilon}{\partial z}}_{9}\Big) - \underbrace{\rho_\omega h(\varepsilon)\frac{\varepsilon^4}{\mu^2}\frac{\partial^2 s^\varepsilon}{\partial t^2}}_{10}$$

- We have the assumption $\lim_{\varepsilon\to 0}\frac{h(\varepsilon)E(\varepsilon)}{\varepsilon} = E_0$, so we have $h(\varepsilon)E(\varepsilon) \backsim \varepsilon$, so **term 7** $\backsim \varepsilon$
- From above we know **term 8** $\backsim 1 = \varepsilon^0$
- **term 9** $\backsim 1 = \varepsilon^0$
- We have $\|s^\varepsilon(t)\|_{L^2(0,L)} \leqslant C\|A\|_v$, so $s^\varepsilon \backsim 1 = \varepsilon^0$, so **term 10** $\backsim 1 = \varepsilon^5$
- *We take care of the terms which have the same order of ε, then* ***term 10*** *will disappear.*

4.

$$F_z = \mu\Big(\underbrace{\frac{\partial v(\varepsilon)_r}{\partial z}}_{11} + \underbrace{\frac{1}{\varepsilon}\frac{\partial v(\varepsilon)_z}{\partial r}}_{12}\Big)$$

- We have $\|v(\varepsilon)_r\|_{L^2(\Omega\times(0,T))} \leqslant C\frac{\varepsilon^3}{\mu}\|A\|_v$,so $\frac{\partial v(\varepsilon)_r}{\partial z} \backsim \varepsilon^3$, so **term 11** $\backsim \varepsilon^3$
- We have $\|v(\varepsilon)_z\|_{L^2(\Omega\times(0,T))} \leqslant C\frac{\varepsilon^2}{\mu}\|A\|_v$, so $\frac{\partial v(\varepsilon)_z}{\partial r} \backsim \varepsilon^2$, so **term 12** $\backsim \varepsilon$
- *We take care of the terms which have the same order of ε, then* ***term 11*** *will disappear.*

5.

$$-\mu\Big(\frac{1}{\varepsilon^2}\frac{\partial^2 v(\varepsilon)_r}{\partial r^2} + \frac{\partial^2 v(\varepsilon)_r}{\partial z^2} + \frac{1}{\varepsilon^2}\frac{1}{r}\frac{\partial v(\varepsilon)_r}{\partial r} - \frac{1}{\varepsilon^2}\frac{v(\varepsilon)_r}{r^2}\Big) + \frac{1}{\varepsilon}\frac{\partial p(\varepsilon)}{\partial r} = 0$$

- $\|v(\varepsilon)_r\|_{L^2(\Omega\times(0,T))} \leqslant C\frac{\varepsilon^3}{\mu}\|A\|_v$ and $\|p(\varepsilon)\|_{L^2(\Omega\times(0,T))} \leqslant C\|A\|_v$, so this equation has the lowest order of ε^{-1}, so we have $\frac{\partial p(\varepsilon)}{\partial r} = 0 \Rightarrow \underbrace{p = p(\varepsilon)(z,t)}_{*1}$

6.

$$-\mu\Big(\frac{1}{\varepsilon^2}\frac{\partial^2 v(\varepsilon)_z}{\partial r^2}+\underbrace{\frac{\partial^2 v(\varepsilon)_z}{\partial z^2}}_{13}+\frac{1}{\varepsilon^2}\frac{1}{r}\frac{\partial v(\varepsilon)_z}{\partial r}\Big)+\frac{\partial p(\varepsilon)}{\partial z}=0$$

- $\|v(\varepsilon)_z\|_{L^2(\Omega\times(0,T))}\leqslant C\frac{\varepsilon^2}{\mu}\|A\|_v$, so only **term 13** is of order ε^2, the others are of order $1=\varepsilon^0$, so we have $\frac{\partial^2 v(\varepsilon)_z}{\partial r^2}+\frac{1}{r}\frac{\partial v(\varepsilon)_z}{\partial r}=\frac{\partial p(\varepsilon)}{\partial z}\Rightarrow \underbrace{\frac{\partial}{\partial r}\Big(r\frac{\partial v(\varepsilon)}{\partial r}\Big)=r\frac{\partial p(\varepsilon)}{\partial z}}_{*2}$

7.

$$div_\varepsilon v(\varepsilon)=\frac{1}{\varepsilon}\frac{\partial v(\varepsilon)_r}{\partial r}+\frac{\partial v(\varepsilon)_z}{\partial z}+\frac{1}{\varepsilon}\frac{v(\varepsilon)_r}{r}=0$$

- All of these terms are of order ε^2, we have $\underbrace{\frac{\partial}{\partial r}(rv(\varepsilon)_r)+\frac{\partial}{\partial z}(rv(\varepsilon)_z)=0}_{*3}$

8.

$$D_\varepsilon(v(\varepsilon))=\begin{pmatrix}\frac{1}{\varepsilon}\frac{\partial v(\varepsilon)_r}{\partial r} & 0 & \frac{1}{2}(\frac{\partial v(\varepsilon)_r}{\partial z}+\frac{1}{\varepsilon}\frac{\partial v(\varepsilon)_z}{\partial r})\\ 0 & \frac{1}{\varepsilon}\frac{v(\varepsilon)_r}{r} & 0\\ \frac{1}{2}(\frac{\partial v(\varepsilon)_r}{\partial z}+\frac{1}{\varepsilon}\frac{\partial v(\varepsilon)_z}{\partial r}) & 0 & \frac{\partial v(\varepsilon)_z}{\partial z}\end{pmatrix}$$

- The lowest order term is $\frac{1}{\varepsilon}\frac{\partial v(\varepsilon)_z}{\partial r}$ which is order of ε. We have

$$D_{\varepsilon\to 0}(v(\varepsilon))=\begin{pmatrix}0 & 0 & \frac{1}{2}\frac{\partial v(\varepsilon)_z}{\partial r}\\ 0 & 0 & 0\\ \frac{1}{2}\frac{\partial v(\varepsilon)_z}{\partial r} & 0 & 0\end{pmatrix}$$

Then we have the following definitions:

$$A^\varepsilon = \begin{pmatrix}\rho_\omega h(\varepsilon)\frac{\partial^2}{\partial t^2}-h(\varepsilon)G(\varepsilon)k(\varepsilon)\frac{\partial^2}{\partial z^2}+\frac{h(\varepsilon)E(\varepsilon)}{1-\sigma^2}\frac{1}{\varepsilon^2R^2} & \frac{h(\varepsilon)E(\varepsilon)}{1-\sigma^2}\frac{\sigma}{\varepsilon R}\frac{\partial}{\partial z}\\ -\frac{h(\varepsilon)E(\varepsilon)}{1-\sigma^2}\frac{\sigma}{\varepsilon R}\frac{\partial}{\partial z} & \rho_\omega h(\varepsilon)\frac{\partial^2}{\partial t^2}-\frac{h(\varepsilon)E(\varepsilon)}{1-\sigma^2}\frac{\partial^2}{\partial z^2}\end{pmatrix}$$

$$U^\varepsilon = \begin{pmatrix}\eta^\varepsilon\\ s^\varepsilon\end{pmatrix}$$

$$B^\varepsilon = \begin{pmatrix}-\mu\Big(\frac{1}{\varepsilon^2}\frac{\partial^2}{\partial r^2}+\frac{\partial^2}{\partial z^2}+\frac{1}{\varepsilon^2}\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{\varepsilon^2}\frac{1}{r^2}\Big) & 0 & \frac{1}{\varepsilon}\frac{\partial}{\partial r}\\ 0 & -\mu\Big(\frac{1}{\varepsilon^2}\frac{\partial^2}{\partial r^2}+\frac{\partial^2}{\partial z^2}+\frac{1}{\varepsilon^2}\frac{1}{r}\frac{\partial}{\partial r}\Big) & \frac{\partial}{\partial z}\\ \frac{1}{\varepsilon}\frac{\partial}{\partial r}+\frac{1}{\varepsilon}\frac{1}{r} & \frac{\partial}{\partial z} & 0\end{pmatrix}$$

$$W^\varepsilon = \begin{pmatrix}v(\varepsilon)_r\\ v(\varepsilon)_z\\ p(\varepsilon)\end{pmatrix}$$

$$F^\varepsilon = \begin{pmatrix} -F_r \\ -F_z \end{pmatrix} = \begin{pmatrix} p(\varepsilon) - 2\mu\frac{1}{\varepsilon}\frac{\partial v(\varepsilon)_r}{\partial r} \\ -\mu(\frac{\partial v(\varepsilon)_r}{\partial z} + \frac{1}{\varepsilon}\frac{\partial v(\varepsilon)_z}{\partial r}) \end{pmatrix} = \underbrace{\begin{pmatrix} -2\mu\frac{1}{\varepsilon}\frac{\partial}{\partial r} & 0 & 1 \\ -\mu\frac{\partial}{\partial z} & -\mu\frac{1}{\varepsilon}\frac{\partial}{\partial r} & 0 \end{pmatrix}}_{F_1^\varepsilon} W^\varepsilon$$

$$A^0 = \begin{pmatrix} -G_0\frac{\partial^2}{\partial z^2} + \frac{E_0}{R^2(1-\sigma^2)} & \frac{E_0\sigma}{R(1-\sigma^2)}\frac{\partial}{\partial z} \\ -\frac{E_0\sigma}{R(1-\sigma^2)}\frac{\partial}{\partial z} & -\frac{E_0}{1-\sigma^2}\frac{\partial^2}{\partial z^2} \end{pmatrix}$$

$$U^0 = \begin{pmatrix} \eta \\ s \end{pmatrix}$$

$$B^0 = \begin{pmatrix} 0 & 0 & \frac{\partial}{\partial r} \\ 0 & -\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right) & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial r} + \frac{1}{r} & \frac{\partial}{\partial z} & 0 \end{pmatrix}$$

$$W^0 = \begin{pmatrix} v_r \\ v_z \\ p \end{pmatrix}$$

$$F^0 = \begin{pmatrix} p \\ -\frac{\partial v(\varepsilon)_z}{\partial r} \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 0 & 1 \\ 0 & -\frac{\partial}{\partial r} & 0 \end{pmatrix}}_{F_1^0} W^0$$

Now we can present the main convergence theory in a very transparent form!

Theorem 4 *We consider the rescaled fluid-structure interaction problem matrix form*

$$\underbrace{\begin{pmatrix} A^\varepsilon & 0 \\ 0 & B^\varepsilon \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} U^\varepsilon \\ W^\varepsilon \end{pmatrix}}_{X^\varepsilon} = \underbrace{\begin{pmatrix} F^\varepsilon \\ 0 \end{pmatrix}}_{Y^\varepsilon} \tag{6.1}$$

with $F^\varepsilon = F_1^\varepsilon W^\varepsilon$.
The limit problem

$$\underbrace{\begin{pmatrix} A^0 & 0 \\ 0 & B^0 \end{pmatrix}}_{B} \underbrace{\begin{pmatrix} U^0 \\ W^0 \end{pmatrix}}_{X^0} = \underbrace{\begin{pmatrix} F^0 \\ 0 \end{pmatrix}}_{Y^0} \tag{6.2}$$

with $F^0 = F_1^0 W^0$.
The solution $\begin{pmatrix} U^\varepsilon \\ W^\varepsilon \end{pmatrix}$ *to (6.1) converges weakly to the solution* $\begin{pmatrix} U^0 \\ W^0 \end{pmatrix}$ *which is a weak solution of (6.2) in* $\mathcal{V}$ *as* $\varepsilon \to 0$.

Proof :

We show that $U^\varepsilon \rightharpoonup U^0$ *and* $W^\varepsilon \rightharpoonup W^0$ *as* $\varepsilon \to 0$.

$$
\begin{aligned}
\|F^\varepsilon - F^0\| &= \|A^\varepsilon \cdot U^\varepsilon - A^0 \cdot U^0\| \\
&= \|A^\varepsilon \cdot U^\varepsilon - A^\varepsilon \cdot U^0 + A^\varepsilon \cdot U^0 - A^0 \cdot U^0\| \\
&= \|A^\varepsilon \cdot (U^\varepsilon - U^0) + (A^\varepsilon - A^0) \cdot U^0\| \\
&\geq \|A^\varepsilon\|\|U^\varepsilon - U^0\| - \|A^\varepsilon - A^0\|\|U^0\| \\
\Rightarrow & \quad \|A^\varepsilon\|\|U^\varepsilon - U^0\| \leqslant \|F^\varepsilon - F^0\| + \|A^\varepsilon - A^0\|\|U^0\|
\end{aligned}
$$

since we know that $F^\varepsilon \rightharpoonup F^0$ and $A^\varepsilon \rightharpoonup A^0$ as $\varepsilon \to 0$, and $\|B^\varepsilon\|$ and $\|U^0\|$ are bounded, so we have

$$U^\varepsilon \rightharpoonup U^0, \quad as \quad \varepsilon \to 0$$

$$
\begin{aligned}
0 &= \|B^\varepsilon \cdot W^\varepsilon - B^0 \cdot W^0\| \\
&= \|B^\varepsilon \cdot W^\varepsilon - B^\varepsilon \cdot W^0 + B^\varepsilon \cdot W^0 - B^0 \cdot W^0\| \\
&= \|B^\varepsilon \cdot (W^\varepsilon - W^0) + (B^\varepsilon - B^0) \cdot W^0\| \\
&\geq \|B^\varepsilon\|\|W^\varepsilon - W^0\| - \|B^\varepsilon - B^0\|\|W^0\| \\
&\Rightarrow \|B^\varepsilon\|\|W^\varepsilon - W^0\| \leqslant \|B^\varepsilon - B^0\|\|W^0\|
\end{aligned}
$$

since we know that $B^\varepsilon \rightharpoonup B^0$ as $\varepsilon \rightharpoonup 0$, and $\|B^\varepsilon\|$ and W^0 are bounded, so we have

$$W^\varepsilon \rightharpoonup W^0, \quad as \quad \varepsilon \to 0$$

6.1 The limit Problem P($\varepsilon \to 0$)

Actually here the limit problem is identical to the problem (5.49) which is well-defined and has a unique solution p.

We call the domain in the limit situation is Ω_0, the lateral wall Σ_0

For the lateral wall we have the equation

$$A^0 \cdot U^0 = F^0, \tag{6.3}$$

For the fluid we have the equation

$$B^0 \cdot W^0 = 0, \quad in \quad \Omega_0 \times (0, T) \tag{6.4}$$

We have rescaled $D_\varepsilon(\varphi)$ in the limit situation call it $D_0(\varphi)$ which is define as following, for $\varphi = \varphi_r \vec{e}_r + \varphi_z \vec{e}_z$:

$$D_0(\varphi) = \begin{pmatrix} 0 & 0 & \frac{1}{2}\frac{\partial \varphi_z}{\partial r} \\ 0 & 0 & 0 \\ \frac{1}{2}\frac{\partial \varphi_z}{\partial r} & 0 & 0 \end{pmatrix}$$

Proposition 4 *For $P(\varepsilon \to 0)$, we have the following initial-boundary conditions:*

$$
\begin{aligned}
&v_r(R,z,t) = \frac{\partial \eta}{\partial t}, \qquad for\ (z,t) \in ((0,L)\times(0,T))\\
&v_z(R,z,t) = \frac{\partial s}{\partial t}, \qquad for\ (z,t) \in ((0,L)\times(0,T))\\
&\eta = s = \frac{\partial \eta}{\partial t} = \frac{\partial s}{\partial t} = 0, \quad for\ t=0\\
&\frac{\partial s}{\partial z} = \eta = 0,\ p=0, \quad for\ z=0,\ t\in(0,T)\\
&s=\eta=0,\ p=A(t), \quad for\ z=L,\ t\in(0,T)
\end{aligned}
$$

Motivated by the weak formulation of the rescaled problem $P(\varepsilon)$, assumptions and the compactness result, we define the following weak formulation for $P(\varepsilon \to 0)$. Define :

$$
\begin{aligned}
\boldsymbol{\varepsilon}_{fluid\to 0}(v,\varphi,\psi;0) &= 2\int_0^T\int_{\Omega_0}(D_0(v):D_0(\varphi))\psi(t)rdrdzdt\\
\boldsymbol{\varepsilon}_{elastic\to 0}(\eta,s,\varphi,\psi;0) &= R\int_0^T\int_0^L\left\{G_0\frac{\partial\eta}{\partial z}\frac{\partial\varphi_r}{\partial z}+\frac{E_0}{1-\sigma^2}\left(\frac{\sigma}{R}\frac{\partial s}{\partial z}+\frac{\eta}{R^2}\right)\varphi_r\right.\\
&\quad\left.+\frac{E_0}{1-\sigma^2}\left(\frac{\partial s}{\partial z}\frac{\partial\varphi_z}{\partial z}-\frac{\sigma}{R}\frac{\partial\eta}{\partial z}\varphi_z\right)\right\}\psi(t)dzdt\\
\boldsymbol{\varepsilon}_{pressure\to 0}(A,\varphi,\psi) &= \int_0^T\int_0^R A(t)\varphi_z\psi(t)rdrdt
\end{aligned}
$$

Definition 8 *(Weak formulation of the limiting problem $P(\varepsilon \to 0)$). Let $G_0 > 0.(v,p,\eta,s)$ is a solution if following conditions are satisfied*

$$
\boldsymbol{\varepsilon}_{fluid\to 0}(v,\varphi,\psi;0)+\boldsymbol{\varepsilon}_{elastic\to 0}(\eta,s,\varphi,\psi;0) = -\boldsymbol{\varepsilon}_{pressure\to 0}(A,\varphi,\psi), in\ C(\mathbb{R}_+), \forall\varphi\in V
$$

From the expression of $\boldsymbol{\varepsilon}_{fluid\to 0}(v,\varphi,\psi;0)$ and $D_0(\varphi)$, we have for $P(\varepsilon \to 0)$

$$
v_r = 0
$$

Chapter 7

Modelling Blood Flow in the Compliant Tube using COMSOL Multiphysics

This section is too much to do actually, but limited by our time, we can not go that far. So far as what I understand, **COMSOL Multiphysics** is a very powerful tool to simulate things and you can use **COMSOL Script** to implement your things if you can not find the things you needed in application modes. You can generate your own graphical user interface(GUI) for an application generated in **COMSOL Script**, or for a **COMSOL Multiphysics** model. And a user-defined GUI is implemented in an m-file creating a graphical environment that uses predefined JAVA components such as menu bars, tabs, panels, buttons, check boxes, axes, tables, etc. To generate a GUI you first need to create a frame. This is the main window for the application. To the frame you can add menus or panels. In the panels you add different graphical components such as labels, text fields, buttons, combo boxes or even other panels.

7.1 Introduction

Here we show how to use **COMSOL Multiphysics** to model fluid-structure interaction problem. It illustrates how fluid flow can deform the surrounding structure as well as how to solve the flow in a continuously deforming geometry using the Arbitrary Lagrangian-Eulerian (ALE) technique.

7.2 Model Definition

Here, the compliant tube has a length L(m) which we can change and the inner radius $r = 0.004m$, the wall thickness is $h = 4e-4m$, with the density $\rho_\omega = 1.1kg/m^2$, the Young modulus is $E = 6000$Pa, and the shear modulus is $G * k = 500000$Pa.

The fluid is incompressible, and we assume here the blood is an incompressible fluid even only in some situation we can have this assumption, further, with the density $\rho = 1050kg/m^3$ and dynamic viscosity $\mu = 3.4e - 3Pa \cdot s$.

The model consists of a fluid part, solved by Incompressible Navier-Stokes equations in the compliant tube, and a structural mechanics part, which will be solved by Navier equations in our model, but inside of Multiphysics, we just use what they provided, namely is **Solid,Stress-Strain (transient analysis)**. Since here we have a compliant structural part, so it is reasonable to use **ALE(Arbitrary Lagrangian Eulerian)** method. And this Moving Mesh application mode makes sure the fluid-structural interaction movement which means that the fluid will deform the lateral wall and have the displacements which will change the domain of the fluid. For the equations, first for the fluid to structure,the fluid solution provides the values of one term which is function of the fluid stresses at the wall, for the structure and fluid, the movement of the vessel wall changes the geometry on which the fluid equations must be solved. In addition, the proper boundary conditions for the fluid velocity in correspondence to vessel wall are not anymore homogeneous ,but they impose the equality between the flluid and the structure belocity. They express the fact that the fluid particle in correspondence of the vessel wall should move at the same velocity as the wall.

The model accounts for transient effects in both the fluid and the structure. **It models the structural deformation using large deformations in the Plane Strain application mode**. The displacements and its velocities are denoted by $\mathbf{u}, \mathbf{v}, \mathbf{u_t}, \mathbf{v_t}$. The Stokes equations describe the fluid flow, where the velocity components and pressure are denoted by $\mathbf{u_2}, \mathbf{v_2}$ and $\mathbf{p_2}$.

7.3 Fluid Flow

We describe fluid flow in the compliant tube with Incompressible Navier-Stokes equations,sloving the velocity field, $\mathbf{u} = (u, v)$ and the pressure p in the spacial (deformed) moving coordinate system :

$$\rho\frac{\partial \mathbf{u}}{\partial t} + \rho(\mathbf{u} \cdot \nabla)\mathbf{u} = \nabla \cdot [-p\mathbf{I} + \mu(\nabla \mathbf{u} + (\nabla \mathbf{u})^T)] + \mathbf{F}$$
$$\nabla \cdot \mathbf{u} = 0$$

where $\mathbf{F}$ is the volume force affecting the fluid. But here, we assume no gravitational or other volume forces affecting the fluid, so $\mathbf{F} = 0$. The coordinate-system velocity is $\mathbf{u}_m = (u_m, v_m, w_m)$.

'Inlet' where $z = 0$, as we call it, we assume the pressure $p = P_0$, on the 'outlet' where $z = L$, we assume the pressure is $p = A(t) + P_0$ where P_0 is the reference pressure, along the lateral wall, there is no no-slip condition imposed. We assume the rate of the deformation is equal to the fluid velocity, $u_2 = \frac{\partial u}{\partial t}$ and $v_2 = \frac{\partial v}{\partial t}$. And the whole system is driven by the pressure difference $A(t)$.

7.4 Structural Mechanics

The model solves for the structural deformations using an elastic formulation and a nonlinear-geometry formulation to allow for large deformations.

The lateral wall experience a load from the fluid given by

$$\mathbf{F}_T = (-p\mathbf{I} + \mu(\nabla\mathbf{u} + (\nabla\mathbf{u})^T))\mathbf{n}$$

where $\mathbf{n}$ is the normal vector to the boundary. This load represents a sum of pressure and viscous forces.

7.5 Discussion

An advantage with a 3D ALE simulation is that it visualizes the actual FSI movement. The drawback is that there are many sources for error accumulation in the numerical algorithm and a rigorous error analysis based on a posteriori error estimation is still an unsolved problem. Also a time discretization with a fine resolution leads to a time consuming costy computation even for a linear approximation with small deformations. Therefore we have focused the study in this thesis on an alternative approach based on asymptotic analysis which yields a reduced limit equation (for the pressure) which preserves the major features of the original FSI problem thanks to the convergence result (Theorem 4). The other important quantities, like velocity and displacement can then be resolved from the reduced equation for the pressure.

7.6 Modelling in COMSOL Multiphysics

This model we use three application mode:

1. using **Incompressible Navier-Stokes** to model the fluid .It is active only inside of the tube.Further,

2. using **Moving Mesh(ALE)** to model the moving boundaries, notice that both **Incompressible Navier-Stokes** and **Moving Mesh(ALE)** are working under frame **ALE**

3. using **Solid,Stress-Strain** to analysis the displacements,stresses and strains that results in a 3D body given applied loads and constraints.

7.7 Modelling Using the Graphical User Interface

MODEL NAVIGATOR

1. In the **Model Navigator** select **3D** from the **Space dimension** list and click the **Multiphysics** button.

2. From the list of application modes select **COMSOL Multiphysics>Fluid Dynamics>Incompressible Navier-Stokes> Transient analysis** , then click **Add**

3. Under **Application Mode Properties** set **Weak constraints** to **Non-ideal** and set **Frame** to **Frame(ALE)**.

4. Under **COMSOL Multiphysics>Structural Mechanics>Solid,Stress-Strain> Transient analysis** , then click **Add**

5. Under **Application Mode Properties**, set **Large deformation** to **On**, set **Frame** to **Frame(xyz)** and set **Weak constraints** to **Non-ideal**.

6. Under **COMSOL Multiphysics>Deformed Mesh(ALE)> Moving Mesh(ALE)> Transient analysis** , then click **Add**

7. Under **Application Mode Properties** set **Smoothing method** to **Winslow** and set **Defines frame** to **Frame(ALE)** and **Motion relative to:** to **Frame(xyz)** and set **Weak constraints** to **Non-ideal**.

8. click **OK**

GEOMETRY MODELLING

1. Select **Axes/Grid Settings** from **Options**,to set the axis, uncheck **Auto** box,set both **r spacing** and **z spacing** to 0.002,then click**OK** .

2. select **Specify Objects¿Circle** from **Draw**,set **Radius** to 0.004, then click **OK**.

3. then repeat above step to draw a circle whose radius is 0.004-(4e-4).And choose both circles using **Ctrl+A**.

4. select **Extrude** from **Draw**,set **Distance** to be the length of the tube L.

PHYSICS SETTINGS

1. From the **Options** menu choose **Constants**

2. In the **Constants** dialog box define the following names and expressions, after this,click **OK**.

SUBDOMAIN SETTINGS

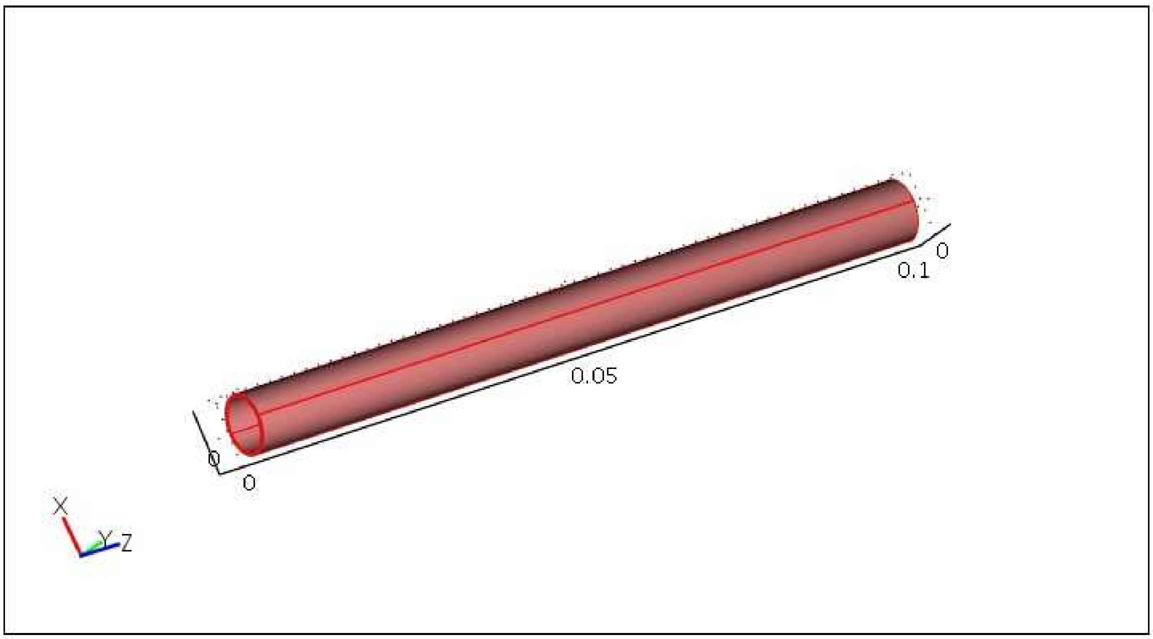


Figure 7.1: the blood vessel with headlight

1. From the **Multiphysics** menu, make sure **Incompressible Navier-Stokes(ns)** is selected.

2. Under **Physics** menu, select **Subdomain Settings** and in the dialog box select Subdomain **2** and set the density and dynamic viscosity of your material and corresponding volume force in x,y,z directions.Here we do not have any volume force, so set all to zero.

3. From the **Multiphysics** menu, make sure **Solid,Stress-Strain(smsld)** is selected.

4. Under **Physics** menu, select **Subdomain Settings** and in the dialog box select Subdomain **1** and set all **quantity**.

5. From the **Multiphysics** menu, make sure **Moving Mesh(ALE)** is selected.

6. Under **Physics** menu, select **Subdomain Settings** and in the dialog box select Subdomain **1** and under **Mesh**,check **Physics induced displacement** and set them to **u,v,w** separately, and select Subdomain **2**,check the box **Free displacement**

BOUNDARY SETTINGS

1. From the **Multiphysics** menu, make sure **Incompressible Navier-Stokes(ns)** is selected.

2. Set boundary **5,6,10,11** which are the lateral boundary to **Slip/Symmetry** and the inlet **7**,set **Outflow/Pressure**,to zero and the outlet **8**,set the pressure you want,mainly because our system is driven by the pressure drop.

3. From the **Multiphysics** menu, make sure **Solid,Stress-Strain(smsld)** is selected.

4. Only set boundary **R**of three directions x,y,z of the inlet and the outlet between the layer to zero

5. From the **Multiphysics** menu, make sure **Moving Mesh(ALE)** is selected.

6. set **mesh displacement** of the boundaries inlet and outlet **7,8** to zero and set the others to **u,v,w** in x,y,z directions respectively.

MESH

Here you can try different mesh generating methods depending on how fast and how accurate you want by changing values in **Mesh parameters**. Here, we consider the efficency, so we did not use so fine meshes, otherwise it will take a long time to run.

SOLVE

This is the last step. and you just click the **Solve** button on the menu.

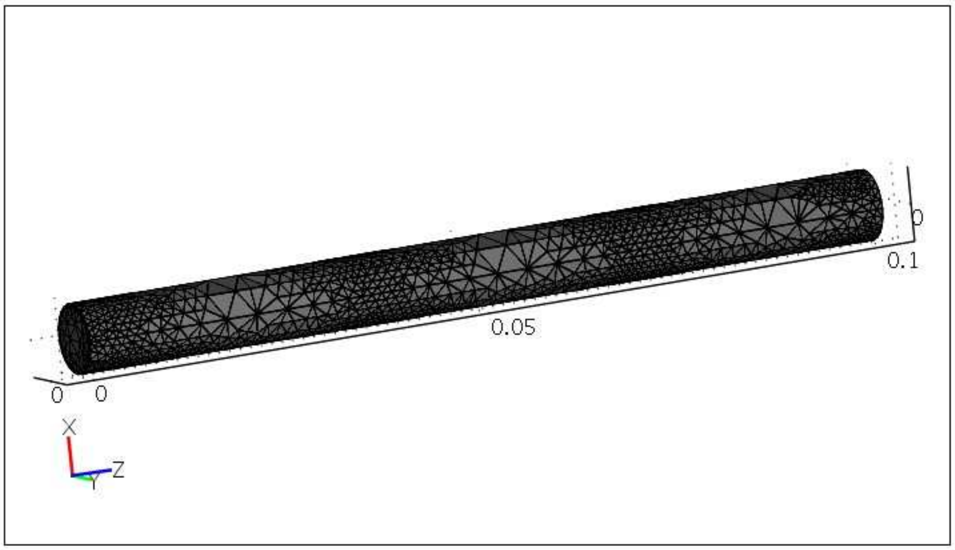


Figure 7.2: the mesh

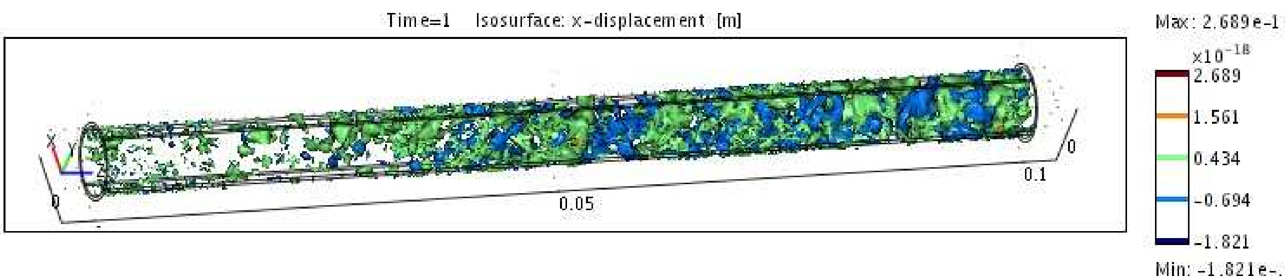


Figure 7.3: Isosurface plot after using Incompressible Navier-Stokes equations to model the fluid.

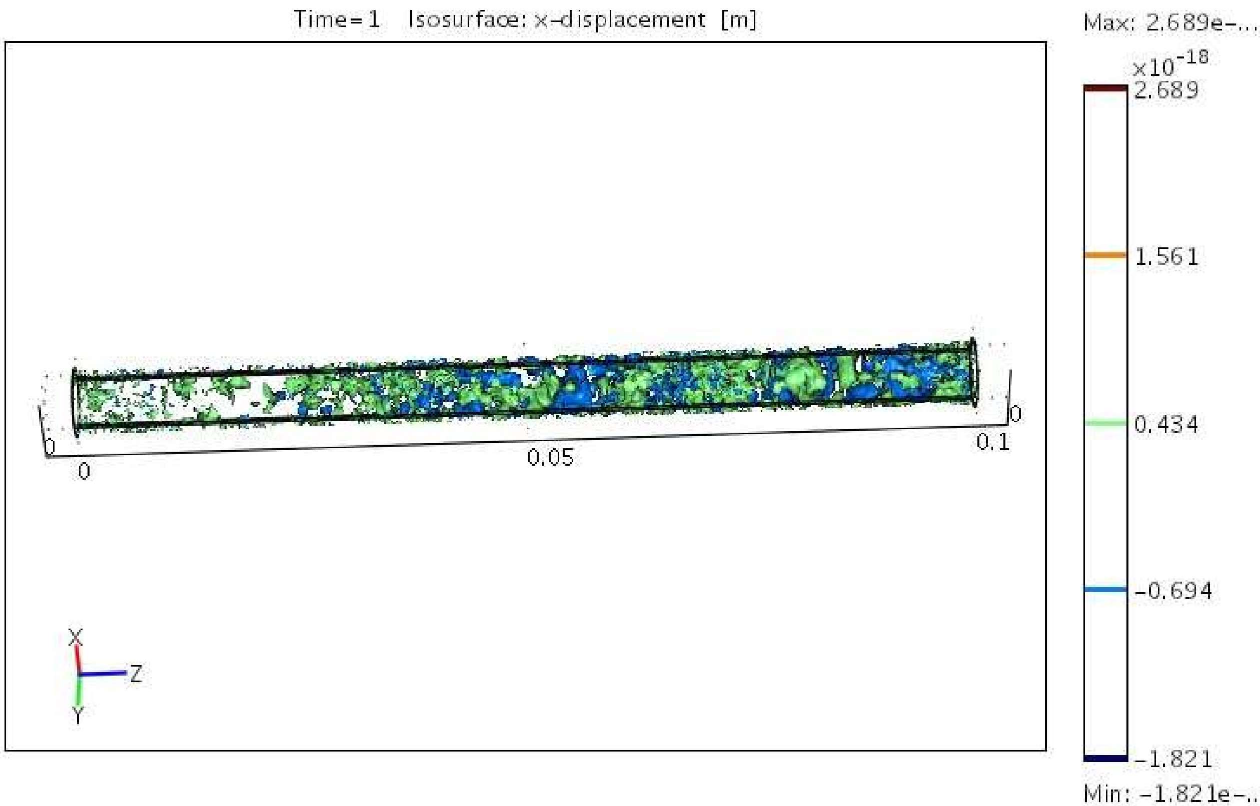


Figure 7.4: Isosurface plot after using Stokes equation to model the fluid.

Chapter 8

Appendix

8.1 Gronwall Inequality

Let f be a non-negative function which is integrable in $I = (t_0, t_1)$ and g, ϕ be two continuous functions in I, with g nondecreasing. If

$$\phi(t) \leqslant g(t) + \int_{t_0}^{t} f(\tau)\phi(\tau)d\tau, \qquad \forall t \in I$$

Then

$$\phi(t) \leqslant g(t)exp(\int_{t_0}^{t} f(\tau)d\tau), \qquad \forall t \in I$$

8.2 Poincaré Inequality-multidimensional case

Let $f : \mathbb{R}^N \to \mathbb{R}$ be a function in $H^1(\Omega)$, with f=0 on $\Gamma \subset \partial\Omega$ of strictly positive measure. Then, there exists a positive constants C_p (depending only on the domain Ω and on Γ), such that

$$\|f\|_{L_2(\Omega)} \leqslant C_p \|\nabla f\|_{L^2(\Omega)}$$

8.3 Navier-Stokes equations

$$\begin{aligned} &\rho(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u}) = -\nabla p + F + \eta \Delta \mathbf{u}, ---Momentum \qquad equation \\ &\nabla \cdot \mathbf{u} = 0. \qquad --------------Continuity \qquad equation \end{aligned}$$

where $\mathbf{u}$ is the velocity of the fluid, p is the pressure, F is the body force, ρ is the density of the fluid, η is the dynamic viscosity, $\nu = \frac{\eta}{\rho}$ is the kinematic viscosity. In the following we will give the equations under **cylindrical coordinates** with the components of the velocity vector given by $\mathbf{u} = (u_r, u_\theta, u_z)$, the *continuity equation* is:

$$\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0$$

and the Navier-Stokes equations are given by

$$\rho\Big(\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta} + u_z\frac{\partial u_r}{\partial z} - \frac{u_\theta^2}{r}\Big)$$
$$= -\frac{\partial p}{\partial r} + \eta\Big(\frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r}\frac{\partial u_r}{\partial r} - \frac{u_r}{r^2} + \frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2} + \frac{\partial^2 u_r}{\partial z^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}\Big) + F_r$$

$$\rho\Big(\frac{\partial u_\theta}{\partial t} + u_r\frac{\partial u_\theta}{\partial r} + u_r\frac{u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta} + u_z\frac{\partial u_\theta}{\partial z}\Big)$$
$$= -\frac{1}{r}\frac{\partial p}{\partial \theta} + \eta\Big(\frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r^2} + \frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2} + \frac{\partial^2 u_\theta}{\partial z^2} - \frac{2}{r^2}\frac{\partial u_r}{\partial \theta}\Big) + F_\theta$$

$$\rho\Big(\frac{\partial u_z}{\partial t} + u_r\frac{\partial u_z}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta} + u_z\frac{\partial u_z}{\partial z}\Big)$$
$$= -\frac{\partial p}{\partial z} + \eta\Big(\frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r}\frac{\partial u_z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2} + \frac{\partial^2 u_z}{\partial z^2}\Big) + F_z$$

8.4 Lax-Milgram Lemma

Let ϕ be a bounded, coercive, bilinear functional on a Hilbert space H. For every bounded linear functional f on H, there exists a unique $x_f \in H$ such that

$$f(x) = \phi(x, x_f)$$

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