NONSTANDARD ANALYSIS

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In school mathematics and in university mathematics a straight line is identified with the set of points whose coordinates are real numbers. But there is an argument opposed to such concrete identification. This argument is based on the fact that there are infinitely many properties of a straight line that can neither be proved nor disproved using the Zermelo-Fraenkel axioms of set theory.

Gottfried Leibniz (1646-1716) looked differently at a straight line. He viewed it as a carrier of sets of points larger than the set of reals, sets including ideal infinitely small elements greater than zero and smaller than any real number. The so-called Leibniz principle authorized the application to infinitesimals of operations applied to real numbers. Thus multiplication by -1 yielded negative elements, addition of infinitesimals to reals resulted in the insertion of new numbers between reals, and division of 1 by positive infinitesimals yielded infinite numbers, numbers greater than any real number.

[Leibniz introduced the symbol df/dt for the derivative and the symbol $\int f(x)dx$ for the integral ($\int$ was an extended version of the letter S). These symbols were a reminder that the derivative differed from the differential quotient—a quotient of infinitesimals—by an infinitesimal, and that, similarly, the integral differed from an appropriate sum by an infinitesimal.]

The Leibniz approach contributed in large measure to the flourishing of analysis in the 18th century and was reflected in the works of Leonhard Euler (1707–1783), the greatest mathematician of that time. But a modern reader of Euler's works is certain to realize that Leibniz's principle cannot be used without appropriate restrictions. In fact, its implicit contradictions were criticized from the very beginning. Its most coherent and famous critic was bishop George Berkeley (1685–1783), who referred to infinitesimals as "ghosts of departed quantities."

In the 19th century the method of infinitesimals was gradually replaced by the since dominant epsilon–delta method. By the end of the 19th century the theory had reached a high level of development and—somewhat paradoxically—strengthened the resistance to the use of infinitesimals of many generations of mathematicians who believed Cantor's claim (1845–1918) that one could use set theory to prove the nonexistence of infinitesimals. The slow renaissance of infinitesimal analysis began with Thoralf Skolem's (1887–1963) nonarchimedean model of arithmetic involving infinite numbers (1933 and 1934).

[An archimedean model of arithmetic is a model in which for every number N and for every positive number ϵ there is a finite number n such that

$$\epsilon + \epsilon + \ldots + \epsilon > N,$$

where ϵ is taken n times. To use picture language: an archimedean model has the property that no matter how small the steps we take we eventually reach any specific number.]

The Polish mathematician Jerzy Łoś (1920–1998) made an important advance by constructing the hyperreal numbers as the closed extension (containing infinitesimals) of the ordered field of reals. Łoś also gave a modern and rigorous version of Leibniz's principle, called the transfer principle, that states precisely which assertions about real numbers carry over to infinitesimals. But the final step on this road was taken by Abraham Robinson (1918–1974), creator of nonstandard analysis. Robinson showed that the new version of the Leibniz principle makes possible a development of analysis based on the hyperreal numbers.

The usefulness of nonstandard analysis is based on two of its properties: the transfer principle—mentioned earlier—and a property called saturation, which reflects the richness of the system of hyperreal numbers. The first application of nonstandard analysis was the filling in of the reasoning gaps in the 18th-century calculus of infinitesimals. In addition, nonstandard analysis yielded a new model of a number system in which the real numbers are a subset of the hyperreal numbers. Actually, nonstandard analysis goes much further, in that it supplies models for classical mathematics based on either the real or the hyperreal number systems and provides the possibility of carrying out all classical arguments in each of these two structures. Furthermore, objects of the real structure can be interpreted within the framework of a hyperreal structure in a new and fruitful way. In that structure, the transfer principle provides a sharp criterion that singles out sets to which one can apply Leibniz's principle. For example, one can apply this principle to the set of natural numbers smaller than a given natural number but not to the set of all finite natural numbers. The reason for the latter restriction is that by taking the least upper bound of this set (a step applicable to all bounded sets of real numbers) we would obtain a nonexistent entity, namely, the smallest infinite integer.

The most important aspect of nonstandard analysis is its role as a powerful tool for contemporary and future mathematics. When it comes to applications of mathematics to natural phenomena and to economics, the hyperreal straight line and nonstandard analysis furnish far more models than the classical real structure. Physicists often substitute infinite sets for finite sets of atoms or particles. Nonstandard analysis goes much further, in that it enables one to substitute for a finite set a set whose cardinality is that of a definite infinite integer. The principles of finite combinatorics retain their validity in nonstandard analysis. For example, it is easy to compute the probability of events of interest to physicists.

The author of the present article is interested in the study of rarified gases (an area of mathematical physics involving complicated equations) in which forces—say, gravitational forces between gas particles—approach infinity and the stabilization time of the induced motion becomes infinite. The power of nonstandard analysis with reference to infinite magnitudes has often enabled me to find solutions of the gas equations and to study the limiting behavior of a gas when time tends to infinity. When I made use of ideas connected with the transfer principle, my ideas acquired significance and were interpretable within the framework of the traditional approach, but when using classical methods it was often extremely difficult to obtain any initial results.

The great mathematician Kurt Gödel predicted that nonstandard analysis would be the mathematics of the 21st century. But now, in 2004, nonstandard analysis is not commonly used by mathematicians. This is so because contemporary mathematicians are usually specialists, and few of them are equally comfortable in the realm of mathematical logic and in the realm of applications. Given its tremendous power, nonstandard analysis undoubtedly deserves a place among the fundamental methods to be used by mathematicians of future generations.

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