Anonym kod	**LMA019 Algebra 2017-10-26**	Poäng

1. Till nedanstående uppgifter skall korta lösningar redovisas, samt svar anges, på anvisad plats (endast lösningar och svar på detta blad, och på anvisad plats, beaktas).

   (a) Bestäm en vektor som har längden 1 och är vinkelrät mot de två vektorerna $(1, 2, 3)$ och $(2, -1, 1)$. (3 p)

   Lösning:

$$\mathbb{v} = \begin{vmatrix} \overset{+}{i} & \overset{-}{j} & \overset{+}{k} \\ 1 & 2 & 3 \\ 2 & -1 & 1 \end{vmatrix} = \begin{pmatrix} 5 \\ 5 \\ -5 \end{pmatrix}$$

$$|\mathbb{v}| = \sqrt{25+25+25} = \sqrt{3 \cdot 25} = 5\sqrt{3}$$

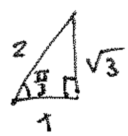
$$\frac{1}{|\mathbb{v}|}\mathbb{v} = \frac{1}{5\sqrt{3}}(5, 5, -5) = \frac{1}{\sqrt{3}}(1, 1, -1)$$

   Svar: $\frac{1}{\sqrt{3}}(1, 1, -1)$

   (b) Beräkna $\sin\left(-\frac{\pi}{12}\right)$ (3 p)

   Lösning:

$$\sin\left(-\frac{\pi}{12}\right) = -\sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right) =$$

$$= -\left(\sin\frac{\pi}{3}\cos\frac{\pi}{4} - \sin\frac{\pi}{4}\cos\frac{\pi}{3}\right) =$$ {  } =

$$= -\left(\frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\cdot\frac{1}{2}\right) = \frac{1-\sqrt{3}}{2\sqrt{2}}$$

   Svar: $\frac{1-\sqrt{3}}{2\sqrt{2}}$

   (c) För vilket eller vilka värden på $a \in \mathbb{R}$ ligger vektorerna $\mathbf{v}_1 = (1, 1, 2)$, $\mathbf{v}_2 = (4, 6, 1)$ och $\mathbf{v}_3 = (5, a, -11)$ i samma plan? (3 p)

   Lösning:

$$\mathbb{n} = \begin{vmatrix} i & j & k \\ 1 & 1 & 2 \\ 4 & 6 & 1 \end{vmatrix} = \begin{pmatrix} -11 \\ 7 \\ 2 \end{pmatrix}$$

$\mathbb{v}_1, \mathbb{v}_2, \mathbb{v}_3$ i samma plan $\Leftrightarrow \mathbb{n} \perp \mathbb{v}_3 \Leftrightarrow \mathbb{n} \cdot \mathbb{v}_3 = 0$

$\Leftrightarrow -55 + 7a - 22 = 0 \Leftrightarrow 7a = 77 \Leftrightarrow a = 11$

   Svar: $a = 11$

(d) Beräkna determinanten (3 p)

$$\begin{vmatrix} 1 & 2 & -1 & -2 \\ 2 & 3 & 0 & -1 \\ 1 & 2 & 1 & 4 \\ 1 & 3 & -1 & 0 \end{vmatrix} \begin{matrix} -2 & -1 \end{matrix} =$$

Lösning:

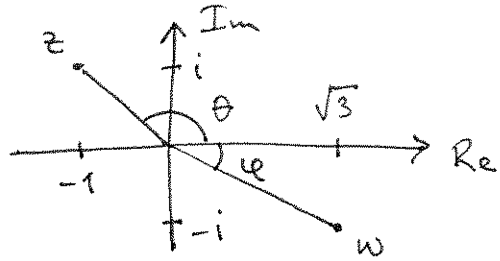
$$= \begin{vmatrix} 1 & 2 & -1 & -2 \\ 0 & -1 & 2 & 3 \\ 0 & 0 & 2 & 6 \\ 0 & 1 & 0 & 2 \end{vmatrix} \begin{matrix} 1 \end{matrix} = \begin{vmatrix} 1 & 2 & -1 & -2 \\ 0 & -1 & 2 & 3 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 2 & 5 \end{vmatrix} \begin{matrix} -1 \end{matrix} =$$

$$= \begin{vmatrix} 1 & 2 & -1 & -2 \\ 0 & -1 & 2 & 3 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & -1 \end{vmatrix} = 1 \cdot (-1) \cdot 2 \cdot (-1) = 2$$

Svar: 2

(e) Förenkla det komplexa talet $\dfrac{(-1+i)^{27}}{(\sqrt{3}-i)^{18}}$ så långt som möjligt. (3 p)

Lösning: Låt $z = -1+i$, $w = \sqrt{3}-i$



$|z| = \sqrt{2}$, $|w| = 2$

$\theta = \frac{\pi}{2} + \frac{\pi}{4} = \frac{3\pi}{4}$, $\varphi = -\frac{\pi}{6}$

$$\frac{z^{27}}{w^{18}} = \frac{\left(\sqrt{2}\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right)\right)^{27}}{\left(2\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)\right)^{18}} = \{\text{de Moivre}\} =$$

$$= \frac{2^{27/2}\left(\cos\left(\frac{81\pi}{4}\right) + i\sin\left(\frac{81\pi}{4}\right)\right)}{2^{18}\left(\cos\left(+\frac{18\pi}{6}\right) - i\sin\left(\frac{18\pi}{6}\right)\right)} = 2^{27/2-18}\,\frac{\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)}{\cos(\pi) - i\sin(\pi)} =$$

$$= 2^{-5}\cdot\sqrt{2}\cdot\frac{1/\sqrt{2} + i/\sqrt{2}}{-1} = -2^{-5}(1+i) = -\frac{1}{32} - i\frac{1}{32}$$

Svar: $-\frac{1}{32} - i\frac{1}{32}$

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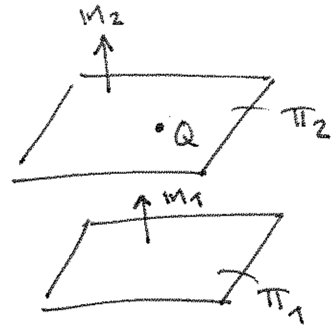


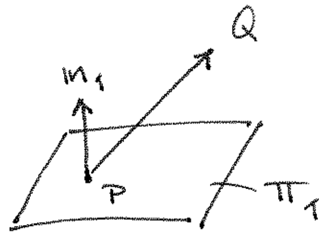
2. (a) $\Pi_1 \parallel \Pi_2 \iff n_1 \parallel n_2 \Rightarrow$

$\Rightarrow n_2 = n_1 = (1,-1,1)$ (längden av n_2 irrelevant)

$\Rightarrow \Pi_2 : x - y + z = D$

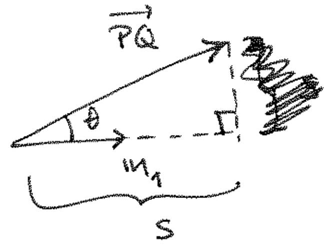
$Q \in \Pi_2 \Rightarrow D = 3 - 1 + 3 = 5$

$\therefore \Pi_2 : x - y + z = 5$



(b) $P = (1,0,0) \in \Pi_1$

$\Rightarrow \overrightarrow{PQ} = Q - P = (2,1,3)$



$$s = |\overrightarrow{PQ}|\cos\theta = \frac{|\overrightarrow{PQ}||n_1|\cos\theta}{|n_1|} =$$

$$= \frac{\overrightarrow{PQ} \cdot n_1}{|n_1|}$$

$\therefore$ Minsta avståndet $= |s| = \dfrac{|\overrightarrow{PQ} \cdot n_1|}{|n_1|} =$

$$= \frac{|(2,1,3)\cdot(1,-1,1)|}{|(1,-1,1)|} = \frac{|2-1+3|}{\sqrt{3}} = \frac{4}{\sqrt{3}} \text{ l.e.}$$

3 (a) <u>Definition</u>: En funktion $T : \mathbb{R}^n \to \mathbb{R}^m$ sägs vara en <u>linjär</u> <u>avbildning</u> om

(i) $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ för alla $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$

(ii) $T(c\mathbf{u}) = cT(\mathbf{u})$ för alla $c \in \mathbb{R}$, $\mathbf{u} \in \mathbb{R}^n$

(b) Om $T(\mathbf{x}) = A\mathbf{x}$ så $A = \big(T(\mathbf{e}_1)\ T(\mathbf{e}_2)\ T(\mathbf{e}_3)\big)$

Vet redan $T(\mathbf{e}_1)$ och $T(\mathbf{e}_2)$. Söker $T(\mathbf{e}_3)$

$$T(\mathbf{e}_3) = T\left(\begin{pmatrix}1\\0\\1\end{pmatrix} - \begin{pmatrix}1\\0\\0\end{pmatrix}\right) \overset{(i)}{=} T\begin{pmatrix}1\\0\\1\end{pmatrix} + T\left(-\begin{pmatrix}1\\0\\0\end{pmatrix}\right) \overset{(ii)}{=}$$

$$\overset{(ii)}{=} T\begin{pmatrix}1\\0\\1\end{pmatrix} - T\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}2\\-1\end{pmatrix} - \begin{pmatrix}2\\3\end{pmatrix} = \begin{pmatrix}0\\-4\end{pmatrix}$$

$$\therefore\ A = \big(T(\mathbf{e}_1)\ T(\mathbf{e}_2)\ T(\mathbf{e}_3)\big) = \begin{pmatrix}2 & 0 & 0\\3 & 1 & -4\end{pmatrix}$$

4. Om $y = kx + m$ så

$$\begin{cases} -k + m = 2 \\ 0 + m = 3 \\ 2k + m = 4 \\ 3k + m = 5 \end{cases} \iff \underbrace{\begin{pmatrix} -1 & 1 \\ 0 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} k \\ m \end{pmatrix}}_{\mathbf{x}} = \underbrace{\begin{pmatrix} 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}}_{\mathbf{b}}$$

$$A^T A = \begin{pmatrix} -1 & 0 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 0 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 14 & 4 \\ 4 & 4 \end{pmatrix}$$

$$A^T \mathbf{b} = \begin{pmatrix} -1 & 0 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 21 \\ 14 \end{pmatrix}$$

$$A^T A \mathbf{x} = A^T \mathbf{b} \Rightarrow \left(\begin{array}{cc|c} 14 & 4 & 21 \\ 4 & 4 & 14 \end{array}\right) \cdot \tfrac{1}{2} \sim \begin{pmatrix} 2 & 2 & 7 \\ 14 & 4 & 21 \end{pmatrix} (-7) \sim$$

$$\sim \begin{pmatrix} 2 & 2 & 7 \\ 0 & -10 & -28 \end{pmatrix} \cdot 5 \sim \begin{pmatrix} 10 & 10 & 35 \\ 0 & -10 & -28 \end{pmatrix} (1) \sim \begin{pmatrix} 10 & 0 & 7 \\ 0 & -10 & -28 \end{pmatrix} \begin{matrix} \cdot \frac{1}{10} \\ \cdot (-\frac{1}{10}) \end{matrix} \sim$$

$$\sim \begin{pmatrix} 1 & 0 & 7/10 \\ 0 & 1 & 28/10 \end{pmatrix} \Rightarrow \begin{pmatrix} k \\ m \end{pmatrix} = \begin{pmatrix} 7/10 \\ 14/5 \end{pmatrix}$$

$$\therefore \quad y = \frac{7}{10} x + \frac{14}{5}$$

5. $AX + B = BX + I \iff AX - BX = I - B \iff$

$\iff (A-B)X = I - B \iff (A-B)^{-1}(A-B)X = (A-B)^{-1}(I-B)$

$\therefore\ X = (A-B)^{-1}(I-B)$

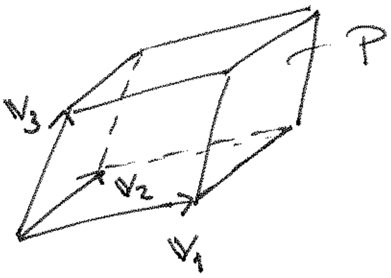
$I - B = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}$

$A - B = \begin{pmatrix} 1 & 1 \\ 3 & 2 \end{pmatrix} \Rightarrow$

$\Rightarrow (A-B)^{-1} = \dfrac{1}{2-3}\begin{pmatrix} 2 & -1 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 3 & -1 \end{pmatrix}$

$\therefore\ X = \begin{pmatrix} -2 & 1 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 3 & -2 \end{pmatrix}$

6. Vet att om $A = (\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3)$



så $\text{Volym}(P) = |\det(A)|$

$$A = (\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3) = \begin{pmatrix} 1 & -1 & -3 \\ -1 & 2 & 0 \\ -3 & 2 & 13 \end{pmatrix}$$

$$\Rightarrow \det(A) = \begin{vmatrix} 1 & -1 & -3 \\ -1 & 2 & 0 \\ -3 & 2 & 13 \end{vmatrix} \begin{matrix} 1 \quad 3 \\ \\ \\ \end{matrix} = \begin{vmatrix} 1 & -1 & -3 \\ 0 & 1 & -3 \\ 0 & -1 & 4 \end{vmatrix} =$$

$$= 1 \cdot \begin{vmatrix} 1 & -3 \\ -1 & 4 \end{vmatrix} = 4 - (-1)\cdot(-3) = 1$$

$$\therefore \text{Volym}(P) = |\det(A)| = |1| = 1 \text{ v.e.}$$

7. (a) T surjektiv $\Rightarrow$ A har en pivot-position i varje rad $\Rightarrow$ $\{A$ 3x3-matris$\}$ $\Rightarrow$ A har en pivot-position i varje kolumn $\Rightarrow$ T injektiv

$\therefore$ Sant

(b) $\bar{z} = \dfrac{3}{z} \Leftrightarrow \bar{z}z = 3$

Om $z = x + iy$ så $\bar{z} = x - iy$ $\Rightarrow$

$\Rightarrow \bar{z}z = (x-iy)(x+iy) = x^2 - (iy)^2 = x^2 + y^2 = |z|^2$

$\therefore \bar{z}z = 3 \Leftrightarrow |z|^2 = 3 \Leftrightarrow |z| = \sqrt{3} < 2$

$\therefore$ Sant

(c) Om $\det(A) \neq 0$ så existerar A^{-1}

$AB = AC \Leftrightarrow A^{-1}AB = A^{-1}AC \Leftrightarrow B = C$

$\therefore$ Sant

(d) $A = (\mathbb{a}_1\ \mathbb{a}_2\ \mathbb{a}_3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\mathbb{b} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$

I så fall har $A\mathbf{x} = \mathbb{b}$ en unik lösn. men kolumn. i A är linjärt beroende

$\therefore$ Falskt

(e) $A^2 = A \Rightarrow \det(A^2) = \det(A) \Leftrightarrow \det(A \cdot A) = \det(A)$

$\Leftrightarrow (\det(A))^2 = \det(A) \Leftrightarrow \det(A)(\det(A) - 1) = 0$

$\Rightarrow \det(A) = 1$ eller $\det(A) = 0$

$\therefore$ Falskt

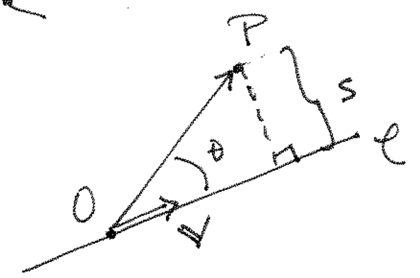
8. (a) <u>Definition</u>: Mängden av alla linjärkombinationer

$$c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \dots + c_m \mathbf{v}_m \qquad c_1, \dots, c_m \in \mathbb{R}$$

av en mängd vektorer $\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$ kallas för det <u>linjära</u> <u>höljet</u> av $\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$

(b) $\text{Span}\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}\right\}$ = rät linje, ℓ, genom origo, O, med riktningsvektor $\mathbf{v} = \begin{pmatrix}1\\1\\1\end{pmatrix}$, dvs

$$\ell : \mathbf{x} = \mathbf{0} + t\mathbf{v} = t(1,1,1), \quad t \in \mathbb{R}$$



$$s = |\overrightarrow{OP}| \sin\theta = \frac{|\overrightarrow{OP}||\mathbf{v}|\sin\theta}{|\mathbf{v}|} = \frac{|\overrightarrow{OP} \times \mathbf{v}|}{|\mathbf{v}|}$$

$$\overrightarrow{OP} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 4 & 5 \\ 1 & 1 & 1 \end{vmatrix} = (-1, 2, -1)$$

$$\Longrightarrow s = \frac{\sqrt{1+4+1}}{\sqrt{1+1+1}} = \frac{\sqrt{6}}{\sqrt{3}} = \sqrt{\frac{6}{3}} = \sqrt{2} \text{ l.e.}$$

9. (a) $VL = \tan\left(\frac{x}{2}\right) = \frac{\sin(x/2)}{\cos(x/2)} = \frac{\sin(x/2)\cos(x/2)}{\cos^2(x/2)} =$

$= \frac{2\sin(x/2)\cos(x/2)}{2\cos^2(x/2)} = \left\{ \begin{array}{l} \text{Vet att:} \qquad \text{Låt } t = \frac{x}{2} \\ \sin(2t) = 2\sin(t)\cos(t), \quad 2\cos^2(t) = 1 + \cos(2t) \end{array} \right\}$

$= \frac{\sin(x)}{1+\cos(x)} = HL$ ☐

(b) $\tan^2\left(\frac{x}{2}\right) = \left(\tan\left(\frac{x}{2}\right)\right)^2 \overset{(a)}{=} \left(\frac{\sin(x)}{1+\cos(x)}\right)^2 =$

$= \frac{\sin^2(x)}{(1+\cos(x))^2} = \{\text{trig. ettan}\} = \frac{1-\cos^2(x)}{(1+\cos(x))^2} =$

$= \frac{(1-\cos(x))(1+\cos(x))}{(1+\cos(x))^2} = \frac{1-\cos(x)}{1+\cos(x)}$ ☐

10 (a) $\mathbf{u} \in \mathbb{R}^n \Rightarrow \mathbf{u} = (u_1, \ldots, u_n)$

$$VL = |\mathbf{u}|^2 = \left(\sqrt{u_1^2 + \ldots + u_n^2}\right)^2 = u_1^2 + \ldots + u_n^2$$

$$HL = \mathbf{u} \cdot \mathbf{u} = (u_1, \ldots, u_n) \cdot (u_1, \ldots, u_n) = u_1^2 + \ldots + u_n^2 \quad \blacksquare$$

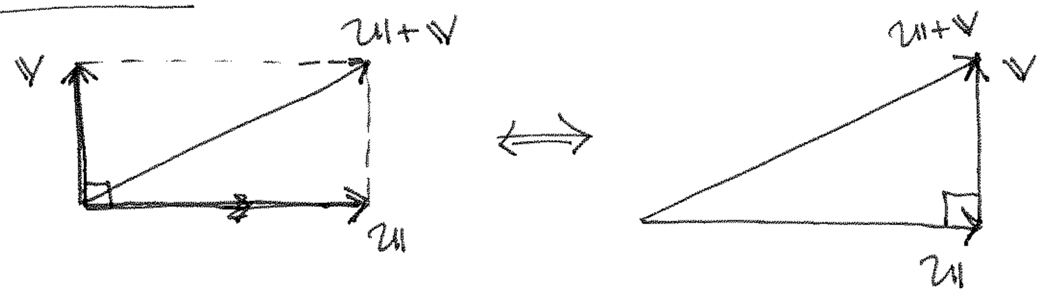
(b) $|\mathbf{u} + \mathbf{v}|^2 \overset{(a)}{=} (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) =$

$$= \mathbf{u} \cdot \mathbf{u} + \mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} =$$

$$= \{ \mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u} = 0 \} =$$

$$= \mathbf{u} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} \overset{(a)}{=} |\mathbf{u}|^2 + |\mathbf{v}|^2 \quad \blacksquare$$

<u>$n = 2$ eller 3:</u>



$\therefore$ Pythagoras sats