Anonym kod	LMA019 Algebra 2017-12-09	Poäng

1. Till nedanstående uppgifter skall korta lösningar redovisas, samt svar anges, på anvisad plats (endast lösningar och svar på detta blad, och på anvisad plats, beaktas).

(a) Bestäm en ekvation för det plan som innehåller de tre punkterna $A = (1, 2, 3)$, $B = (-2, 3, 0)$ och $C = (5, 0, -1)$. (3 p)

Lösning: $\overrightarrow{AB} = B - A = (-3, 1, -3)$, $\overrightarrow{AC} = C - A = (4, -2, -4)$

$$\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} +\hat{i} & -\hat{j} & +\hat{k} \\ -3 & 1 & -3 \\ 4 & -2 & -4 \end{vmatrix} = \begin{pmatrix} -10 \\ -24 \\ 2 \end{pmatrix} \Rightarrow -10x - 24y + 2z = D$$

$$\underline{A \in \pi:} \quad -10 - 48 + 6 = D \Leftrightarrow D = -52$$

$$\therefore \pi: -10x - 24y + 2z = -52 \Leftrightarrow 5x + 12y - z = 26$$

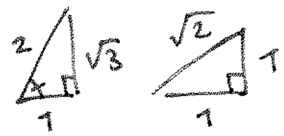
Svar: $5x + 12y - z = 26$

(b) Beräkna $\sin\left(\frac{7\pi}{12}\right)$ (3 p)

Lösning:

$$\sin\left(\frac{7\pi}{12}\right) = \sin\left(\frac{\pi}{3} + \frac{\pi}{4}\right) =$$

$$= \sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{3}\right) = \left\{ \quad \right\} =$$

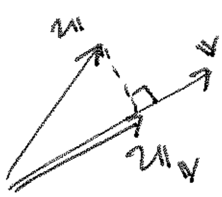


$$= \frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \cdot \frac{1}{2} = \frac{1 + \sqrt{3}}{2\sqrt{2}}$$

Svar: $\frac{1 + \sqrt{3}}{2\sqrt{2}}$

(c) Beräkna vektorprojektionen, $\mathbf{u}_\mathbf{v}$, av $\mathbf{u} = (7, 2, -2)$ ~~på~~ längs $\mathbf{v} = (1, 1, 1)$. (2 p)

Lösning:



$$\mathbf{u}_\mathbf{v} = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2} \mathbf{v} = \frac{(7, 2, -2) \cdot (1, 1, 1)}{(\sqrt{1^2 + 1^2 + 1^2})^2} (1, 1, 1) =$$

$$= \frac{7}{3}(1, 1, 1) = \left(\frac{7}{3}, \frac{7}{3}, \frac{7}{3}\right)$$

Svar: $\left(\frac{7}{3}, \frac{7}{3}, \frac{7}{3}\right)$

(d) Beräkna determinanten av B^{-1} då (3 p)

$$B = \begin{pmatrix} 1 & 0 & 3 & 2 \\ 0 & 0 & 2 & 1 \\ 0 & 2 & 1 & 3 \\ 0 & 2 & 3 & 1 \end{pmatrix}$$

Lösning: $B \cdot B^{-1} = I \Rightarrow \det(B \cdot B^{-1}) = \det(I) \Leftrightarrow \det(B) \cdot \det(B^{-1}) = 1$

$\Leftrightarrow \det(B^{-1}) = \dfrac{1}{\det(B)}$

$$\det(B) = 1 \cdot \begin{vmatrix} 0 & 2 & 1 \\ 2 & 1 & 3 \\ 2 & 3 & 1 \end{vmatrix} = 0 - 2 \cdot (2-6) + 1 \cdot (6-2) =$$

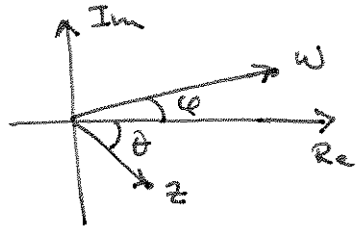
$$= 8 + 4 = 12$$

$$\Rightarrow \det(B^{-1}) = \frac{1}{\det(B)} = \frac{1}{12}$$

Svar: $1/12$

(e) Förenkla det komplexa talet $\dfrac{(\frac{1}{2} - i\frac{\sqrt{3}}{2})^{21}}{(3 + i\sqrt{3})^{48}}$ så långt som möjligt. (3 p)

Lösning: Låt $z = \frac{1}{2} - i\frac{\sqrt{3}}{2}$, $w = 3 + i\sqrt{3}$



$$\theta = -\arctan\left(\frac{\sqrt{3}}{2} / \frac{1}{2}\right) = -\arctan(\sqrt{3}) = -\frac{\pi}{3}$$

$$\varphi = \arctan\left(\frac{\sqrt{3}}{3}\right) = \arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$$

$$|z| = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1, \quad |w| = \sqrt{9+3} = \sqrt{12}$$

$$\Rightarrow \frac{z^{21}}{w^{48}} = \frac{1^{21} \cdot \left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)^{21}}{(\sqrt{12})^{48}\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right)^{48}} = \{\text{de Moivre}\} =$$

$$= \frac{\cos(7\pi) - i\sin(7\pi)}{12^{24}(\cos(8\pi) + i\sin(8\pi))} = \frac{1}{12^{24}} \cdot \frac{\overbrace{\cos(\pi)}^{=-1} - i\overbrace{\sin(\pi)}^{=0}}{\underbrace{\cos(0)}_{=1} + i\underbrace{\sin(0)}_{=0}} =$$

$$= -\frac{1}{12^{24}}$$

Svar: $-\dfrac{1}{12^{24}}$

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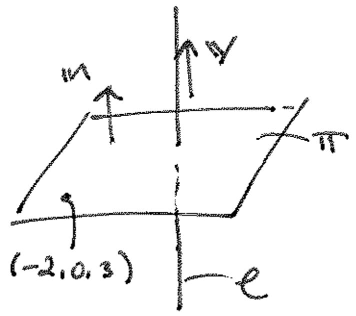
2. (a) $\ell \perp \pi \Leftrightarrow \mathbf{n} \parallel \mathbf{v} \Leftrightarrow$



$\Leftrightarrow \mathbf{n} = \mathbf{v}$ (längden av $\mathbf{n}$ irrelevant!)

$$\ell : \begin{cases} x = 2t + 2 \\ y = t - 7 \\ z = -3t - 6 \end{cases} , \ t \in \mathbb{R} \rightarrow \mathbf{v} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$$

$$\rightarrow \mathbf{n} = (2, 1, -3) \rightarrow$$

$$\Rightarrow \pi : \ 2x + y - 3z = D$$

$$(-2, 0, 3) \in \pi \rightarrow D = 2 \cdot (-2) + 0 - 3 \cdot 3 = -13$$

$$\therefore \ \pi : \ 2x + y - 3z = -13$$

(b) $\ell : \begin{cases} x = 2t + 2 \\ y = t - 7 \\ z = -3t - 6 \end{cases}$ Stoppa in dessa i ekv. för π

$$\Rightarrow 2 \cdot (2t + 2) + t - 7 - 3(-3t - 6) = -13$$

$$\Leftrightarrow 4t + 4 + t - 7 + 9t + 18 = -13$$

$$\Leftrightarrow 14t + 15 = -13 \Leftrightarrow 14t = -28 \Leftrightarrow$$

$$\Leftrightarrow t = -\frac{28}{14} = -2$$

$$\Rightarrow \begin{cases} x = 2 \cdot (-2) + 2 = -2 \\ y = -2 - 7 = -9 \\ z = -3(-2) - 6 = 0 \end{cases}$$

$\therefore$ Skärningspunkt: $(-2, -9, 0)$

3. (a) <u>Definition</u>: $T : \mathbb{R}^n \to \mathbb{R}^n$ injektiv om

$$\mathbf{x} \neq \mathbf{y} \Longrightarrow T(\mathbf{x}) \neq T(\mathbf{y}) \text{ för alla } \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$$

Alternativt om

$$T(\mathbf{x}) = T(\mathbf{y}) \Rightarrow \mathbf{x} = \mathbf{y} \text{ för alla } \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$$

(b) $T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix}$, $T\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\3\end{pmatrix} \Rightarrow T(\mathbf{x}) = \underbrace{\begin{pmatrix}1 & 1\\2 & 3\end{pmatrix}}_{A}\mathbf{x}$

T injektiv om A:s kolumner är linjärt oberoende

$\Longleftrightarrow$

T injektiv om $A\mathbf{x} = \mathbf{0}$ endast har den triviala lösningen $\mathbf{x} = \mathbf{0}$

$$A\mathbf{x} = \mathbf{0} \Rightarrow \left(\begin{array}{cc|c}1 & 1 & 0\\2 & 3 & 0\end{array}\right) -2 \sim \left(\begin{array}{ccc}1 & 1 & 0\\0 & 1 & 0\end{array}\right) -1 \sim \left(\begin{array}{cc|c}1 & 0 & 0\\0 & 1 & 0\end{array}\right)$$

$$\Rightarrow \mathbf{x} = \begin{pmatrix}0\\0\end{pmatrix}$$

$\therefore$ T injektiv

4. (a) $A\mathbf{x} = \mathbf{b} \Rightarrow \left(\begin{array}{cc|c} 1 & -2 & 1 \\ 2 & 1 & 2 \\ 0 & 1 & 1 \end{array}\right) \overset{-2}{\sim} \left(\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 5 & 0 \\ 0 & 1 & 1 \end{array}\right) \begin{array}{l} \\ \leftarrow 5x_2 = 0 \\ \leftarrow x_2 = 1 \end{array} \quad ↯$

$A\mathbf{x} = \mathbf{b}$ saknar lösning

(b) <u>Definition</u>: $\mathbf{x}$ <u>minsta</u> <u>kvadratlösning</u> till $A\mathbf{x} = \mathbf{b}$ om $|A\mathbf{x} - \mathbf{b}|$ är minimal

(c) $A^T A = \begin{pmatrix} 1 & 2 & 0 \\ -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 2 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 0 & 6 \end{pmatrix}$

$A^T \mathbf{b} = \begin{pmatrix} 1 & 2 & 0 \\ -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$

$A^T A \mathbf{x} = A^T \mathbf{b} \Rightarrow \left(\begin{array}{cc|c} 5 & 0 & 5 \\ 0 & 6 & 1 \end{array}\right) \begin{array}{l} \cdot \frac{1}{5} \\ \cdot \frac{1}{6} \end{array} \sim \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1/6 \end{array}\right)$

$\therefore\ \mathbf{x} = (1, 1/6)$ minsta kvadratlösn. till $A\mathbf{x} = \mathbf{b}$

5. $AXA^T = B \iff XA^T = A^{-1}B \iff X = A^{-1}B(A^T)^{-1}$

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \Rightarrow A^{-1} = \frac{1}{1-(-1)}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$(A^T)^{-1} = (A^{-1})^T = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$A^{-1}B = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 & 2 \\ -4 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -2 & 2 \end{pmatrix}$$

$$\Rightarrow A^{-1}B(A^T)^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

$$\therefore \quad X = A^{-1}B(A^T)^{-1} = \underline{\underline{\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}}}$$

6. $\det(AB) = \det(A) \cdot \det(B)$

$$\det(A) = \begin{vmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ -1 & 1 & h \end{vmatrix} = 1 \cdot 2 \cdot h = 2h$$

$$\det(B) = \begin{vmatrix} 3 & 1 & 2 \\ 0 & h & 1 \\ 0 & 0 & h \end{vmatrix} = 3 \cdot h \cdot h = 3h^2$$

$$\Rightarrow \det(AB) = \det(A) \cdot \det(B) = 2h \cdot 3h^2 = 6h^3$$

$$\det(AB) = -48 \Leftrightarrow 6h^3 = -48 \Leftrightarrow h^3 = -8$$

$$\Leftrightarrow \underline{\underline{h = -2}}$$

7. (a) Sant

$$A^2B = A\cdot A\cdot B = A\cdot B\cdot A = B\cdot A\cdot A = BA^2$$

(b) Sant

$AB\mathbf{x} = \mathbf{0}$ endast triv. lösn. $\Rightarrow$ (AB):s kolumner linjärt ober. $\Rightarrow$ $\det(AB) \neq 0$ $\Rightarrow$

$\Rightarrow \{\det(AB) = \det(A)\cdot\det(B)\} \Rightarrow$

$\Rightarrow$ $\det(A) \neq 0$ och $\det(B) \neq 0$ $\Rightarrow$

$\Rightarrow$ Både A och B:s kolumner är linj. ober.

(c) Falskt

Motex. $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

(d) Sant

(e) Falskt

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \overbrace{\begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}}^{A} \overbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}^{\mathbf{x}} + \overbrace{\begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}}^{\mathbf{c}} = A\mathbf{x} + \mathbf{c}$$

$$\Rightarrow T(\mathbf{x}+\mathbf{y}) = A\mathbf{x} + A\mathbf{y} + \mathbf{c} \neq T(\mathbf{x}) + T(\mathbf{y}) =$$

$$= A\mathbf{x} + A\mathbf{y} + 2\mathbf{c}$$

8. $\begin{cases} 4x_1 + 2x_2 - 2x_3 = 1 \\ ax_1 + x_2 + 3x_3 = b \\ x_1 + x_2 + 3x_3 = 1 \end{cases} \Leftrightarrow \underbrace{\begin{pmatrix} 4 & 2 & -2 \\ a & 1 & 3 \\ 1 & 1 & 3 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{\mathbf{x}} = \underbrace{\begin{pmatrix} 1 \\ b \\ 1 \end{pmatrix}}_{\mathbf{b}}$

$A\mathbf{x} = \mathbf{b}$ unik lösn. $\Leftrightarrow$ $\det(A) \neq 0$

$$\det(A) = \begin{vmatrix} 4 & 2 & -2 \\ a & 1 & 3 \\ 1 & 1 & 3 \end{vmatrix} = 4\cdot(3-3) - 2(3a-3) - 2(a-1) =$$

$$= -6a + 6 - 2a + 2 = 8 - 8a$$

$\det(A) = 0 \Leftrightarrow a = 1$

$\therefore$ Unik lösn. om $a \neq 1$

<u>$a=1$</u>: $\begin{pmatrix} 4 & 2 & -2 \\ 1 & 1 & 3 \\ 1 & 1 & 3 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 \\ b \\ 1 \end{pmatrix} \Rightarrow \left(\begin{array}{ccc|c} 4 & 2 & -2 & 1 \\ 1 & 1 & 3 & b \\ 1 & 1 & 3 & 1 \end{array}\right) \sim$

$$\sim \begin{pmatrix} 1 & 1 & 3 & 1 \\ 1 & 1 & 3 & b \\ 4 & 2 & -2 & 1 \end{pmatrix} \begin{matrix} -1 \\ \end{matrix} \sim \begin{pmatrix} 1 & 1 & 3 & 1 \\ 0 & 0 & 0 & b-1 \\ 4 & 2 & -2 & 1 \end{pmatrix} \leftarrow 0 = b-1$$

$\therefore$ Saknas lösning då $a=1$, $b \neq 1$

Oändligt många lösn. då $a=1$, $b=1$

9. Vill beräkna $T\begin{pmatrix}1\\0\end{pmatrix}$ och $T\begin{pmatrix}0\\1\end{pmatrix}$.

$$\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}1\\2\end{pmatrix}-\begin{pmatrix}1\\1\end{pmatrix} \Rightarrow T\begin{pmatrix}0\\1\end{pmatrix}=T\left(\begin{pmatrix}1\\2\end{pmatrix}-\begin{pmatrix}1\\1\end{pmatrix}\right)=$$

$$=\{T \text{ linjär}\}=T\begin{pmatrix}1\\2\end{pmatrix}-T\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}5\\3\end{pmatrix}-\begin{pmatrix}4\\2\end{pmatrix}=\begin{pmatrix}1\\1\end{pmatrix}$$

$$T\begin{pmatrix}1\\0\end{pmatrix}=T\left(\begin{pmatrix}1\\1\end{pmatrix}-\begin{pmatrix}0\\1\end{pmatrix}\right)=T\begin{pmatrix}1\\1\end{pmatrix}-T\begin{pmatrix}0\\1\end{pmatrix}=$$

$$=\begin{pmatrix}4\\2\end{pmatrix}-\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}3\\1\end{pmatrix}$$

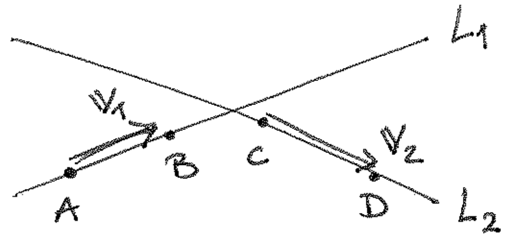
$$\Rightarrow T(\mathbf{x})=\underbrace{\begin{pmatrix}3 & 1\\1 & 1\end{pmatrix}}_{A}\mathbf{x}=A\mathbf{x}\Rightarrow$$

$$\Rightarrow \text{Area}(T(R))=|\det(A)|\cdot \text{Area}(R)=$$

$$=|3-1|\cdot 2\cdot 3=12 \text{ a.e.}$$

10. $\mathbf{v}_1 = \overrightarrow{AB} = B - A = \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$

$\mathbf{v}_2 = \overrightarrow{CD} = D - C = \begin{pmatrix} -6 \\ 6 \\ 2 \end{pmatrix}$



$$\Rightarrow L_1 : \begin{cases} x = 3 - t \\ y = -3 + 2t \\ z = 4 - t \end{cases}, \; t \in \mathbb{R} \qquad L_2 : \begin{cases} x = 3 - 6s \\ y = 6s \\ z = 2 - 2s \end{cases} \quad s \in \mathbb{R}$$

Skärningspunkt om får något s, t :

$$\begin{cases} 3 - t = 3 - 6s \\ -3 + 2t = 6s \\ 4 - t = 2 - 2s \end{cases} \Leftrightarrow \begin{cases} t - 6s = 0 \\ 2t - 6s = 3 \\ t - 2s = 2 \end{cases} \Rightarrow$$

$$\Rightarrow \left(\begin{array}{cc|c} 1 & -6 & 0 \\ 2 & -6 & 3 \\ 1 & -2 & 2 \end{array}\right) \begin{matrix} -2 & -1 \end{matrix} \sim \begin{pmatrix} 1 & -6 & 0 \\ 0 & 6 & 3 \\ 0 & 4 & 2 \end{pmatrix} \begin{matrix} 1 & -\frac{4}{6} \end{matrix} \sim$$

$$\sim \begin{pmatrix} 1 & 0 & 3 \\ 0 & 6 & 3 \\ 0 & 0 & 0 \end{pmatrix} \cdot \frac{1}{6} \sim \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{cases} t = 3 \\ s = 1/2 \end{cases}$$

$t = 3$ insatt i L_1 : $\begin{cases} x = 3 - 3 = 0 \\ y = -3 + 6 = 3 \\ z = 4 - 3 = 1 \end{cases}$

Kontroll : $s = 1/2$ insatt i L_2 : $\begin{cases} x = 3 - 6 \cdot \frac{1}{2} = 0 \\ y = 6 \cdot \frac{1}{2} = 3 \\ z = 2 - 2 \cdot \frac{1}{2} = 1 \end{cases}$ ok !

$\therefore$ Skärningspunkt : $(0, 3, 1)$