Lösningar tentan 20100406

1. a) $f'(x) = \cos(2x) - 2x\sin(2x)$

b) $$f(x) = \frac{\ln(x^2)}{|x|} = \begin{cases} \dfrac{\ln(x^2)}{x} = \dfrac{2\ln x}{x} & \text{om } x>0 \\[2ex] \dfrac{\ln(x^2)}{-x} = \dfrac{\ln((-x)^2)}{-x} = \dfrac{2\ln(-x)}{-x} & \text{om } x<0. \end{cases}$$

$$\Rightarrow f'(x) = \begin{cases} \dfrac{2\left(\frac{1}{x}\cdot x - \ln x\right)}{x^2} = \dfrac{2(1-\ln x)}{x^2} & \text{om } x>0 \\[2ex] \dfrac{2\left(\frac{1}{x}\cdot(-x) - \ln(-x)\cdot(-1)\right)}{x^2} = \dfrac{2(-1+\ln x)}{x^2} & \text{om } x<0. \end{cases}$$

2. $f'(x) = -2x e^{-x^2}$

$f''(x) = -2\left(e^{-x^2} - 2x^2 e^{-x^2}\right) = -2e^{-x^2}(1-2x^2) = 2e^{-x^2}(2x^2-1).$

$f''(x) = 0 \iff 1-2x^2 = 0 \iff x = \pm\frac{1}{\sqrt{2}}$

x		$-\frac{1}{\sqrt{2}}$		$\frac{1}{\sqrt{2}}$	
$f''(x)$	$+$	0	$-$	0	$+$
$f(x)$	$\cup$	infl. pkt	$\cap$	infl pkt	$\cup$

Inflexionspunkter: $\left(\pm\frac{1}{\sqrt{2}}, e^{-1/2}\right)$

3. Skärningspunkten mellan kurvorna : $e^{ax_0} = x_0^2$

Kurvorna tangerar varandra om dem har samma tangent i skärningspunkten. Tangenten i $P(x_0, e^{ax_0})$ har riktningskoefficient $k_1 = e^{ax_0}\cdot a$. Tangenten i $P(x_0, x_0^2)$ har riktningskoefficient $k_2 = 2x_0$. Eftersom $k_1 = k_2 \Rightarrow ae^{ax_0} = 2x_0 \Rightarrow e^{ax_0} = \frac{2x_0}{a}$

$\Rightarrow x_0^2 = e^{ax_0} = \frac{2x_0}{a} \Rightarrow x_0 = 0$ eller $x_0 = \frac{2}{a}$.

Om $x_0 = 0 \Rightarrow e^{ax_0} = e^0 = 1 \neq x_0^2 \Rightarrow$ omöjligt!

Om $x_0 = \frac{2}{a} \Rightarrow ax_0 = 2 \Rightarrow e^{ax_0} = e^2 \Rightarrow x_0^2 = e^2 \Rightarrow x_0 = \pm e \Rightarrow$

$\Rightarrow \frac{2}{a} = \pm e \overset{a>0}{\Longrightarrow} \boxed{a = \frac{2}{e}}.$

(4). $y = \arctan(\sqrt{x})$; $x = 1 \Rightarrow y = \arctan 1 = \frac{\pi}{4}$.

$$y' = \frac{1}{1+(\sqrt{x})^2} \cdot \frac{1}{2\sqrt{x}} = \frac{1}{(1+x)\cdot 2\sqrt{x}} \Rightarrow y'(1) = \frac{1}{4}.$$

Tangentens ekvation är : $y - y(1) = y'(1)(x-1) \Leftrightarrow$

$$\Leftrightarrow y - \frac{\pi}{4} = \frac{1}{4}(x-1) \Leftrightarrow \boxed{y = \frac{x}{4} + \frac{\pi}{4} - \frac{1}{4}}.$$

(5). $x[f(x)]^3 + x f(x) = 4$, $f(2) = 1$.

Derivering ger: $[f(x)]^3 + 3x(f(x))^2 \cdot f'(x) + f(x) + x f'(x) = 0$

Sätt $\left.\begin{array}{l} x = 2 \\ f(2) = 1 \end{array}\right\} \Rightarrow 1 + 3\cdot 2 \cdot 1^2 f'(2) + 1 + 2f'(2) = 0 \Leftrightarrow 8f'(2) + 2 = 0 \Rightarrow$

$$\boxed{f'(2) = -\frac{1}{4}}.$$

(6). $f(x) = x + \frac{\ln x}{x}$; $D_f = (0, +\infty)$.

Asymptoter: $\lim\limits_{x\to 0+}\left(x + \frac{\ln x}{x}\right) = \lim\limits_{x\to 0+} \frac{\ln x}{x} = \frac{-\infty}{0+} = -\infty \Rightarrow x = 0$ lodrät asymptot

$$\lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty}\left(1 + \frac{\ln x}{x^2}\right) = 1 + \lim_{x\to\infty} \frac{\ln x}{x^2} = 1 + 0 = 1 \Rightarrow k = 1.$$

$$\lim_{x\to\infty}(f(x) - x) = \lim_{x\to\infty}\left(x + \frac{\ln x}{x} - x\right) = \lim_{x\to\infty} \frac{\ln x}{x} = 0 \Rightarrow m = 0$$

$\Rightarrow$ linjen $y = x$ är sned asymptot då $x \to +\infty$

$$f'(x) = 1 + \frac{\frac{1}{x}\cdot x - \ln x}{x^2} = 1 + \frac{1 - \ln x}{x^2} = \frac{x^2 - \ln x + 1}{x^2}.$$

$$f''(x) = \frac{-\frac{1}{x}\cdot x^2 + \ln x \cdot 2x}{x^4} = \frac{-x + 2x\ln x}{x^4} = \frac{2\ln x - 1}{x^3}$$

$f''(x) = 0 \Leftrightarrow \ln x = \frac{1}{2} \Leftrightarrow x = e^{1/2}$

Inflexionspunkt $\left(e^{1/2}, e^{1/2} + \frac{1}{2e^{1/2}}\right)$.

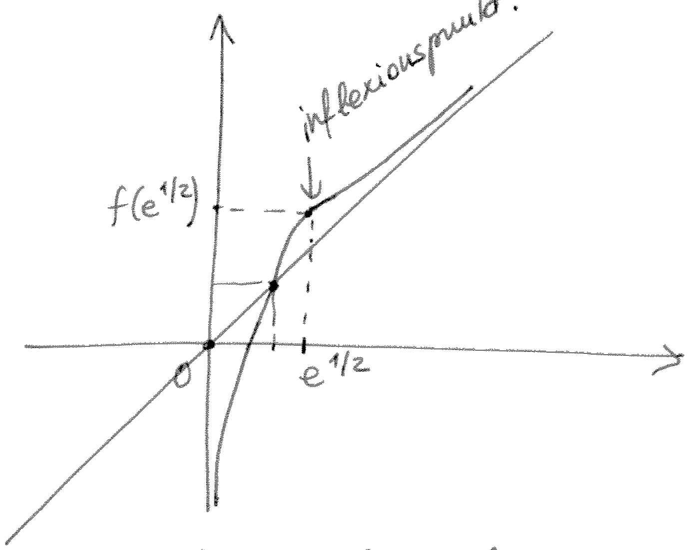
x	0		$e^{1/2}$		$+\infty$
$f''(x)$		$-$	0	$+$	
$f(x)$		$\frown$	infl. punkt	$\smile$	

$f''(x) < 0$ på $(0, e^{1/2}) \Rightarrow f'$ avtagande på $(0, e^{1/2}) \Rightarrow f'(x) > f'(e^{1/2})$

Men $f'(e^{1/2}) = 1 + \frac{1 - \ln(e^{1/2})}{e} = 1 + \frac{1/2}{e} = 1 + \frac{1}{2e} > 0 \Rightarrow f'(x) > 0$ på $(0, e^{1/2})$

$\Rightarrow f$ växande på $(0, e^{1/2})$.

$f''(x) > 0$ på $(e^{1/2}, +\infty) \Rightarrow f'$ växande $\Rightarrow f'(x) > f'(e^{1/2}) > 0 \Rightarrow f$ växande på $(e^{1/2}, \infty)$



$f(x) = x \iff x + \frac{\ln x}{x} = x \iff \ln x = 0$

$\iff x = 1 \Rightarrow (1,1)$ skärning mell. grafen och asymptoten $y = x$.

(7) $D(f \cdot g) = Df \cdot g + f \cdot Dg$

$$\lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} = \lim_{h\to 0} \frac{(f(x+h)-f(x))g(x+h) + f(x)(g(x+h)-g(x))}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \cdot \lim_{h\to 0} g(x+h) + f(x) \cdot \lim_{h\to 0} \frac{g(x+h)-g(x)}{h}$$

$$= f'(x) \cdot g(x) + f(x) \cdot g'(x) \quad \text{ty } \lim_{h\to 0} g(x+h) = g(x) \quad (g \text{ kontinuerlig i } x!)$$

(8). $$\lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} = \lim_{h\to 0} \frac{\sin x \cdot \cos h + \cos x \cdot \sin h - \sin x}{h} =$$

$$= \lim_{h\to 0} \frac{\sin x\,(\cos h - 1)}{h} + \cos x \cdot \underbrace{\lim_{h\to 0} \frac{\sin h}{h}}_{=1} = \sin x \cdot \left(\lim_{h\to 0} \frac{\cos h - 1}{h}\right) + \cos x$$

$$\lim_{h\to 0} \frac{\cos h - 1}{h} = \lim_{h\to 0} \frac{-2\sin^2\frac{h}{2}}{h} = -2 \lim_{h\to 0} \left(\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right)^2 \cdot \frac{h}{4} = 0 \Rightarrow$$

$$\Rightarrow \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} = \sin x \cdot 0 + \cos x = \cos x.$$