Lösningar till Tentan 20110426

① D_f = alla $x \in \mathbb{R}$, $x \neq -2$.

(a) $f(x) = 0 \iff x^2 - 3 = 0 \iff x = \pm\sqrt{3}$

(b) $\lim_{x\to(-2)_-} \frac{x^2-3}{x+2} = \frac{1}{0_-} = -\infty$; $\lim_{x\to(-2)_+} \frac{x^2-3}{x+2} = \frac{1}{0_+} = +\infty$

$\Rightarrow$ $x = -2$ lodrät asymptot (på båda sidorna).

$\lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{x^2-3}{x(x+2)} = \lim_{x\to\infty} \frac{1-\frac{3}{x^2}}{1+\frac{2}{x}} = 1 \Rightarrow k = 1$

$\lim_{x\to\infty} (f(x) - x) = \lim_{x\to\infty} \frac{x^2-3}{x+2} - x = \lim_{x\to\infty} \frac{\cancel{x^2}-3-2x-\cancel{x^2}}{x+2} = -2 \Rightarrow$

$\Rightarrow m = -2 \Rightarrow$ linjen $y = x - 2$ sned asymptot då $x \to \infty$

(samma sak då $x \to -\infty$)

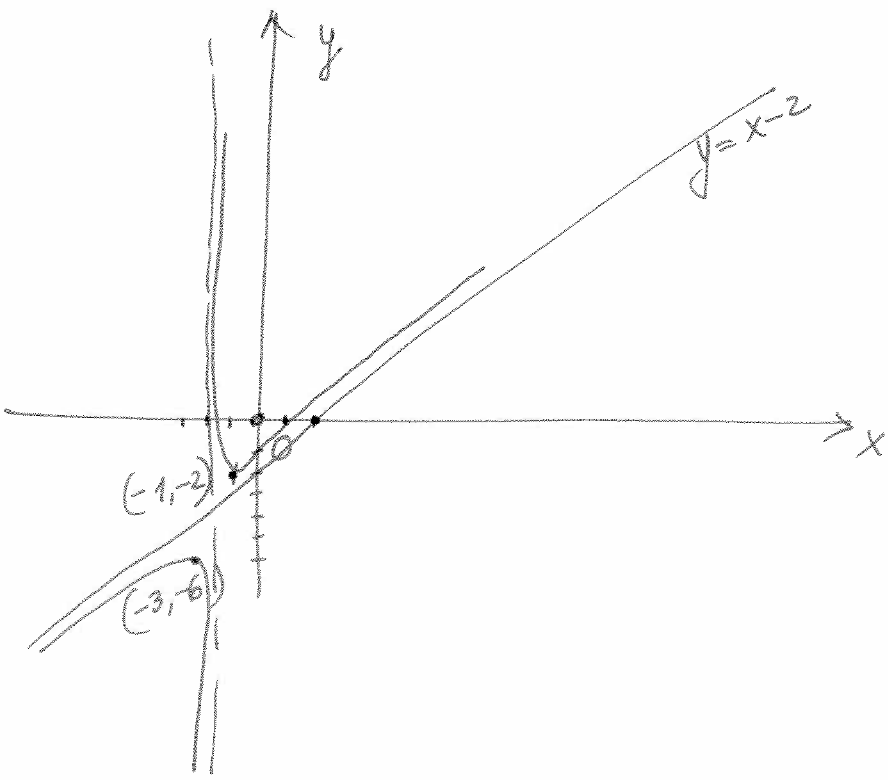
(c) $f'(x) = \frac{2x(x+2)-(x^2-3)}{(x+2)^2} = \frac{2x^2+4x-x^2+3}{(x+2)^2} = \frac{x^2+4x+3}{(x+2)^2}$

$f'(x) = 0 \iff x^2+4x+3 = 0 \iff x = -2 \pm \sqrt{4-3} = -2 \pm 1 \Rightarrow \begin{cases} x_1 = -3 \\ x_2 = -1. \end{cases}$

x		-3		-2		-1	
$f'(x)$	$+$	0	$-$		$-$	0	$+$
$f(x)$	$-\infty \nearrow$	-6 lokal max	$\searrow -\infty$		$+\infty \searrow$	-2 lokal min	$\nearrow +\infty$

$f(-3) = \frac{9-3}{-3+2} = -6$ lokal max

$f(-1) = \frac{1-3}{-1+2} = -2$ lokal min



(2) $y' = e^{\sin x} \cdot \cos x \Rightarrow y'(\pi) = e^{\sin \pi}(\cos \pi) = e^0 \cdot (-1) = -1$

Tangentens ekv : $y - y(\pi) = y'(\pi)(x - \pi)$

$y(\pi) = e^{\sin \pi} = e^0 = 1$

$y - 1 = (-1)(x - \pi) \Leftrightarrow \boxed{y = -x + \pi + 1}$.

(3). $f'(x) = -2xe^{-x^2}$

$f'(x) = 0 \Leftrightarrow x = 0$

x	$-\infty$		0		$+\infty$
$f'(x)$		$+$	0	$-$	
$f(x)$	0	$\nearrow$	1	$\searrow$	0

Största värdet är $f(0) = 1$.

$f''(x) = -2e^{-x^2} - 2x \cdot (-2x)e^{-x^2} = -2e^{-x^2}(1 - 2x^2) = \underbrace{2e^{-x^2}}_{>0}(2x^2 - 1)$

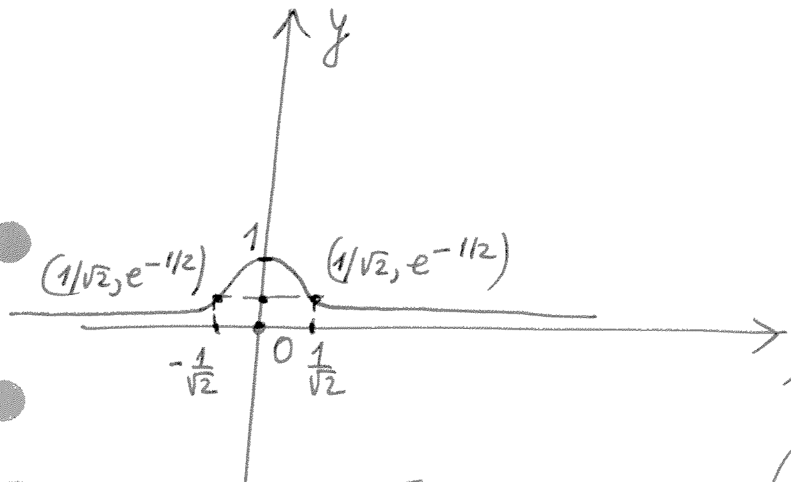
$f''(x) = 0 \Leftrightarrow 1 - 2x^2 = 0 \Leftrightarrow x = \pm \frac{1}{\sqrt{2}}$

x		$-\frac{1}{\sqrt{2}}$		$\frac{1}{\sqrt{2}}$	
$f''(x)$	$+$	0	$-$	0	$+$
$f(x)$	$\smile$	infl	$\frown$	infl.	$\smile$

Inflexionspunkter $\left(\pm \frac{1}{\sqrt{2}}, e^{-1/2}\right)$.

Obs: vågrät asymptot x-axeln.

$\lim_{x \to \pm\infty} e^{-x^2} = 0$.



(4) $\lim_{x \to \infty} \frac{x^3 + e^{2x+1}}{x^3 e^x + e^{2x}} = \lim_{x \to \infty} \left(\frac{\frac{x^3}{e^{2x}} + e}{\frac{x^3}{e^x} + 1} \right) = \frac{e}{1} = \boxed{e}$.

eftersom $\lim_{x \to \infty} \frac{x^3}{e^{2x}} = 0$ och $\lim_{x \to \infty} \frac{x^3}{e^x} = 0$.

⑤ $f'(x) = 2 + \dfrac{(-1)\cdot 2}{(2x+a)^2} = 2 - \dfrac{2}{(2x+a)^2}$

$f'(1) = 0 \Leftrightarrow 2 = \dfrac{2}{(a+2)^2} \Leftrightarrow (a+2)^2 = 1 \Leftrightarrow a+2 = \pm 1 \Rightarrow$

$\Rightarrow a_1 = -3 \;,\; a_2 = -1$

Om $a = -3 \Rightarrow f'(x) = 2 - \dfrac{2}{(2x-3)^2} = \dfrac{2\cdot\left((2x-3)^2 - 1\right)}{(2x-3)^2} = \dfrac{2\cdot(4x^2-12x+8)}{(2x-3)^2} =$

$= \dfrac{8(x^2-3x+2)}{(2x-3)^2} = \dfrac{8(x-1)(x-2)}{(2x-3)^2}.$

		1		3/2		2	
$f'(x)$	+	0	−	⫽	−	0	+
$f(x)$	↗	lokal max	↘	⫽	↘	lokal min	↗

$x=1$ är lokal max inte lokal min!

Så att $a=-3$ stämmer ej!

Om $a = -1 \Rightarrow f'(x) = 2 - \dfrac{2}{(2x-1)^2} = \dfrac{2\cdot(4x^2-4x+1-1)}{(2x-1)^2} = \dfrac{8x(x-1)}{(2x-1)^2}$

		0		1/2		1	
$f'(x)$	+	0	−	⫽	−	0	+
$f(x)$	↗	lokal max	↘	⫽	↘	lokal min	↗

$x=1$ är lokal min

Så att $a=-1$ stämmer!

Svar: $\boxed{a=-1}$.

⑥. D_f : $x>0$ och $\ln x \neq 0 \Leftrightarrow x \neq 1$.

$\lim\limits_{x\to 0+} \dfrac{\ln(2x)}{\ln x} = \lim\limits_{x\to 0+} \dfrac{\ln 2 + \ln x}{\ln x} = \lim\limits_{x\to 0+}\left(1 + \dfrac{\ln 2}{\ln x}\right) = 1$

ty $\lim\limits_{x\to 0+} \ln x = -\infty$ och $\lim\limits_{x\to 0+} \dfrac{\ln 2}{\ln x} = 0$. Ingen asymptot i $x=0$.

$\left.\begin{array}{l} \lim\limits_{x\to 1+} \dfrac{\ln(2x)}{\ln x} = \cancel{\lim} \dfrac{\ln 2}{0+} = +\infty \\ \lim\limits_{x\to 1-} \dfrac{\ln(2x)}{\ln x} = \dfrac{\ln 2}{0-} = -\infty \end{array}\right\} \Rightarrow$ $x=1$ lodrät asymptot

$\lim\limits_{x\to\infty} \dfrac{\ln(2x)}{\ln x} = \lim\limits_{x\to\infty} \dfrac{\ln 2 + \ln x}{\ln x} = \lim\limits_{x\to\infty}\left(\dfrac{\ln 2}{\ln x} + 1\right) = 1$

$\Rightarrow$ $y=1$ är vågrät asymptot då $x\to\infty$.

$$f'(x) = \frac{\frac{2}{2x}\cdot \ln x - \ln(2x)\cdot\frac{1}{x}}{(\ln x)^2} = \frac{\ln x - (\ln 2 + \ln x)}{x(\ln x)^2} = \frac{-\ln 2}{x(\ln x)^2}.$$

$f'(x) = 0 \iff \ln 2 = 0$ omöjligt! $\Rightarrow$ inga nollställen till f'.

x	0		1		$+\infty$
$f'(x)$	∥	$-$	∥	$-$	
$f(x)$	∥ 1	$\searrow$	$-\infty$ ∥ $+\infty$	$\searrow$	1

Inga lokala max/min punkter.

$$f''(x) = \frac{-\ln 2\,(-1)}{x^2(\ln x)^4}\cdot\left((\ln x)^2 + x\cdot 2\ln x\cdot\frac{1}{x}\right) = \frac{\ln 2}{x^2(\ln x)^3}\,(\ln x + 2)$$

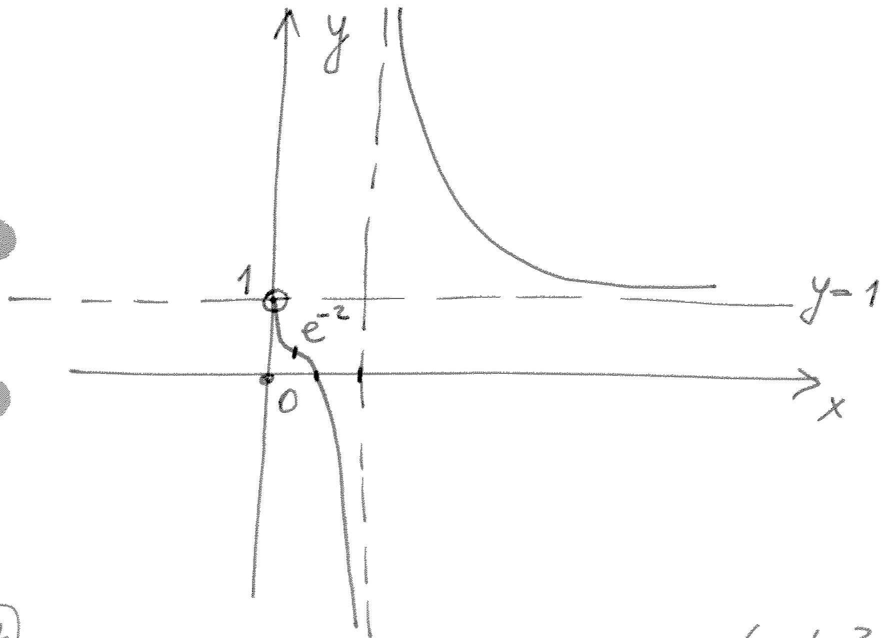
$f''(x) = 0 \iff \ln x + 2 = 0 \iff x = e^{-2}$

x	0		e^{-2}		1		$+\infty$
$f''(x)$	∥	$+$	0	$-$	∥	$+$	
$f(x)$	∥	$\cup$	infl. punkt	$\cap$	∥	$\cup$	

Inflexionspunkt:

$P(e^{-2}, 1 - \frac{\ln 2}{2})$

$f(e^{-2}) = \frac{\ln 2 - 2}{-2} = 1 - \frac{\ln 2}{2}$



(7) $y = \arctan x \Rightarrow x = \tan y \Rightarrow 1 = (1+\tan^2 y)\cdot y' \Rightarrow y' = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}.$

(8) $$\lim_{h\to 0}\frac{f(g(x+h)) - f(g(x))}{h} = \lim_{h\to 0}\frac{f(g(x+h)) - f(g(x))}{g(x+h)-g(x)}\cdot\frac{g(x+h)-g(x)}{h} =$$

$$= \lim_{k\to 0}\frac{f(g(x)+k) - f(g(x))}{k}\cdot\lim_{h\to 0}\frac{g(x+h)-g(x)}{h} = f'(g(x))\cdot g'(x).$$

här är $k = g(x+h) - g(x) \longrightarrow 0$ då $h \to 0$ (eftersom g kontinuerlig).