Lösningar till Tentan 20120305

(1). a) $\lim_{h\to 0}\dfrac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\dfrac{\frac{1}{(x+h)^3}-\frac{1}{x^3}}{h} =$

$= \lim_{h\to 0}\dfrac{x^3-(x+h)^3}{hx^3(x+h)^3} = \lim_{h\to 0}\dfrac{\cancel{x^3}-(\cancel{x^3}+3x^2h+3xh^2+h^3)}{hx^3(x+h)^3} =$

$= \lim_{h\to 0}\dfrac{-\cancel{h}(3x^2+3hx+h^2)}{\cancel{h}x^3(x+h)^3} = \dfrac{-3x^2}{x^3\cdot x^3} = -\dfrac{3}{x^4} \Rightarrow \boxed{f'(x)=-3x^{-4}}$

b) $f(2)=\frac{1}{8} \Rightarrow P(2,\frac{1}{8})$

Tangentens ekv. : $y-\frac{1}{8}=f'(2)(x-2)$

$f'(x)=-3x^{-4} \Rightarrow f'(2)=\dfrac{-3}{2^4}=\dfrac{-3}{16} \Rightarrow$ tangenten har ekv:

$y-\frac{1}{8}=\frac{-3}{16}(x-2) \Leftrightarrow y-\frac{1}{8}=\frac{-3x}{16}+\frac{3}{8} \Leftrightarrow y=-\frac{3x}{16}+\frac{1}{2}.$

Om $y=0 \Rightarrow \frac{3x}{16}=\frac{1}{2} \Leftrightarrow 3x=8 \Rightarrow x=\frac{8}{3} \Rightarrow A(\frac{8}{3},0)$

Om $x=0 \Rightarrow y=\frac{1}{2} \Rightarrow B(0,\frac{1}{2})$

Area$(\Delta OAB) = \frac{1}{2}\cdot OA\cdot OB = \frac{1}{2}\cdot\frac{8}{3}\cdot\frac{1}{2} = \boxed{\frac{2}{3}}.$

(2). $D_f =$ alla $x\in\mathbb{R}$, $x\neq 3$.

$\lim_{x\to 3_-}\dfrac{2x^2-3x-1}{x-3} = \dfrac{18-9-1}{0_-} = \dfrac{8}{0_-} = -\infty$

$\lim_{x\to 3+}\dfrac{2x^2-3x-1}{x-3} = \dfrac{8}{0+} = +\infty$

$\Rightarrow$ $x=3$ lodrät asymptot på båda sidorna

$\lim_{x\to\pm\infty}\dfrac{2x^2-3x-1}{x(x-3)} = \lim_{x\to\pm\infty}\dfrac{2-\frac{3}{x}-\frac{1}{x^2}}{1-\frac{3}{x}} = 2 \Rightarrow k=2.$

$\lim_{x\to\pm\infty}\left(\dfrac{2x^2-3x-1}{x-3}-2x\right) = \lim_{x\to\pm\infty}\dfrac{\cancel{2x^2}-3x-1-\cancel{2x^2}+6x}{x-3} =$

$= \lim_{x\to\pm\infty}\dfrac{3x-1}{x-3} = \lim_{x\to\pm\infty}\dfrac{3-\frac{1}{x}}{1-\frac{3}{x}} = 3 \Rightarrow m=3$

Linjen $y=2x+3$ är sned asymptot då $x\to\pm\infty$

$$f'(x) = \frac{(4x-3)(x-3) - (2x^2-3x-1)\cdot 1}{(x-3)^2} = \frac{4x^2-15x+9-2x^2+3x+1}{(x-3)^2} =$$

$$= \frac{2x^2-12x+10}{(x-3)^2} = \frac{2(x^2-6x+5)}{(x-3)^2}$$

$$f'(x) = 0 \iff x^2-6x+5 = 0 \iff x_{1,2} = 3 \pm \sqrt{9-5} = 3 \pm 2 \Rightarrow \begin{cases} x_1 = 1 \\ x_2 = 5 \end{cases}$$

$$\Rightarrow f'(x) = \frac{2(x-1)(x-5)}{(x-3)^2}.$$

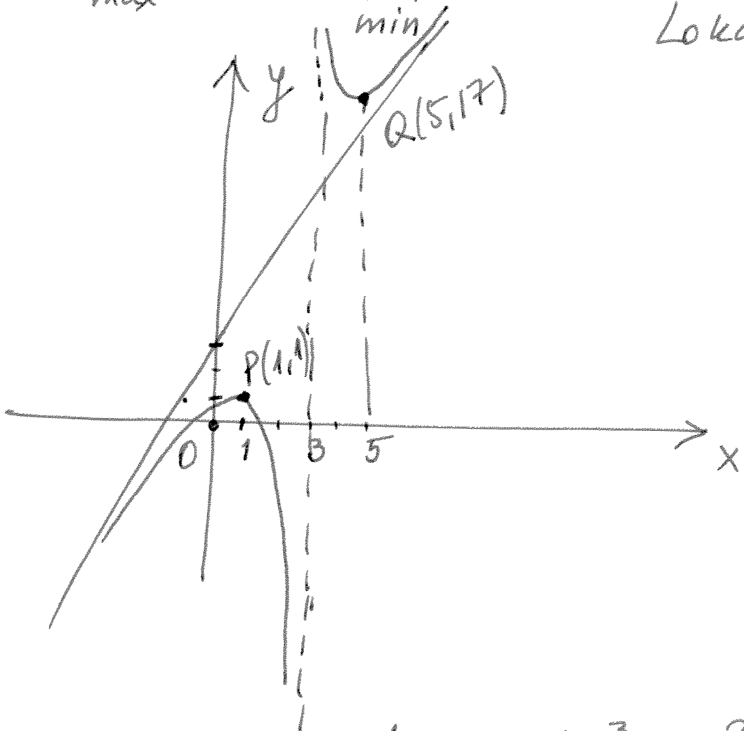
x		1		3		5	
$f'(x)$	+	0	−	//	−	0	+
$f(x)$	↗	lokal max	↘	//	↘	lokal min	↗

$$f(1) = \frac{2-3-1}{1-3} = 1$$

$$f(5) = \frac{2\cdot 25-15-1}{2} = 17$$

Lokal max $P(1,1)$

Lokal min $Q(5,17)$



(3) $D(\sqrt{x^4+1}) = \frac{1}{2\sqrt{x^4+1}}\cdot 4x^3 = \frac{2x^3}{\sqrt{x^4+1}}.$

$D(\sin^2(3x)) = \underbrace{2\sin(3x)\cdot\cos(3x)}_{=\sin(6x)}\cdot 3 = 3\sin(6x).$

$D(x\ln(x^4+1)) = \ln(x^4+1) + x\cdot\frac{1}{x^4+1}\cdot 4x^3 = \ln(x^4+1) + \frac{4x^4}{x^4+1}.$

(4) a) $\lim_{x\to 0}\frac{\sin(3x)}{x} = \lim_{x\to 0}\frac{\sin(3x)}{3x}\cdot 3 = \boxed{3}$ (ty $\lim_{t\to 0}\frac{\sin t}{t} = 1$)

b) $\lim_{x\to\infty}\frac{xe^{2x}}{e^{3x}} = \lim_{x\to\infty}\frac{x}{e^{3x-2x}} = \lim_{x\to\infty}\frac{x}{e^x} = \boxed{0}$

(e^x starkare än polynom)

c) $\lim_{x\to\infty}\left(\ln(2x+1)-\ln(x+3)\right) = \lim_{x\to\infty} \ln\left(\frac{2x+1}{x+3}\right) = \ln\left(\lim_{x\to\infty}\frac{2x+1}{x+3}\right) =$

$= \ln\left(\lim_{x\to\infty}\frac{2+\frac{1}{x}}{1+\frac{3}{x}}\right) = \boxed{\ln 2}.$

5 $\quad xy^3+3y^2-x^2+8=0$

Sätt $y=2 \Rightarrow 8x+12-x^2+8=0 \Leftrightarrow -x^2+8x+20=0 \Leftrightarrow$

$x^2-8x-20=0 \Leftrightarrow x = 4\pm\sqrt{16+20} = 4\pm 6 \Rightarrow x_1=-2,\ x_2=10$

Eftersom man söker punkten i andra kvadranten, $\boxed{x=-2}$.

Punkten är $P(-2,2)$.

Derivering ger: $y^3+x\cdot 3y^2y'+6yy'-2x=0$

Sätt $x=-2,\ y=2 \Rightarrow 8+(-2)\cdot 3\cdot 2^2y'+12y'-2\cdot(-2)=0$

$\Leftrightarrow 12-24y'+12y'=0 \Leftrightarrow 12y'=12 \Rightarrow y'=1 \Rightarrow \boxed{y'(-2)=1}.$

Normalen i $P(-2,2)$ har ekv:

$$y-y(-2) = \frac{-1}{y'(-2)}(x+2) \Leftrightarrow y-2 = \frac{-1}{1}(x+2) \Leftrightarrow$$

$\Leftrightarrow \boxed{y=-x}.$

6. $D_f=\mathbb{R}$. Inga lodrätta asymptoter.

$\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\frac{x}{e^{x^2/2}} = 0$ (ty $e^{x^2/2}$ starkare än $\frac{x^2}{2}$, som är starkare än x).

$\Rightarrow y=0$ är vågrät asymptot, då $x\to\infty$. Samma sak då $x\to-\infty$.

$\lim_{x\to-\infty}\frac{x}{e^{x^2/2}} = \lim_{t\to\infty}\frac{-t}{e^{t^2/2}} = 0.$

$f'(x) = e^{-x^2/2} + xe^{-x^2/2}\cdot\left(\frac{-2x}{2}\right) = e^{-x^2/2}(1-x^2) = \underbrace{e^{-x^2/2}}_{>0}(1-x)(1+x)$

$f'(x)=0 \Leftrightarrow x^2=1 \Leftrightarrow x=\pm 1.$

x	$-\infty$		-1		1		$+\infty$
$f'(x)$		$-$	0	$+$	0	$-$	
$f(x)$	0	$\searrow$	lokal min	$\nearrow$	lokal max	$\searrow$	0

$f(-1) = -e^{-1/2} = \frac{-1}{\sqrt{e}}$ lokal min

$f(1) = e^{-1/2} = \frac{1}{\sqrt{e}}$ lokal max

$$f''(x) = e^{-x^2/2} \cdot (-x)\,(1-x^2) + e^{-x^2/2} \cdot (-2x) = -e^{-x^2/2}\, x\,(3-x^2)$$

$$f''(x) = e^{-x^2/2}\, x\,(x-\sqrt{3})(x+\sqrt{3})$$

$$f''(x) = 0 \iff x_1 = 0,\ x_{2,3} = \pm\sqrt{3}.$$

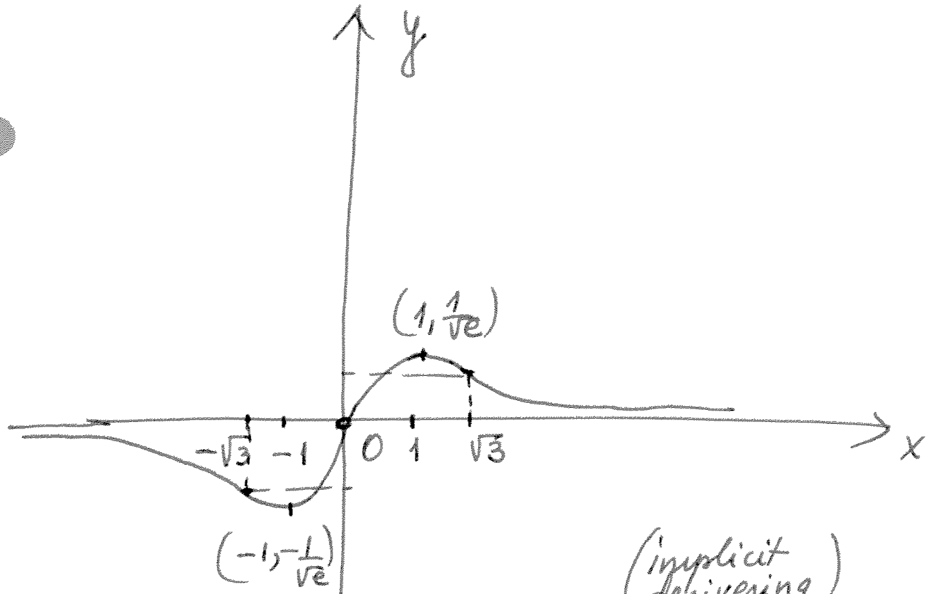
x	$-\infty$		$-\sqrt{3}$		0		$\sqrt{3}$		$+\infty$
$f''(x)$		$-$	0	$+$	0	$-$	0	$+$	
$f(x)$		$\frown$	infl. pkt	$\smile$	infl pkt	$\frown$	infl pkt	$\smile$	

<u>Inflexionspunkter</u>:

$f(0) = 0$

$f(\sqrt{3}) = \sqrt{3}\, e^{-3/2}$

$f(-\sqrt{3}) = -\sqrt{3}\, e^{-3/2}$.



(7) $\arctan x = y \Rightarrow x = \tan y \Rightarrow$ (implicit derivering) $(1+\tan^2 y)\cdot y' = 1 \Rightarrow y' = \dfrac{1}{1+\tan^2 y}$

$\Rightarrow y' = \dfrac{1}{1+x^2}$, för alla reella x.

(8) $$\lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} = \lim_{h\to 0} \frac{(f(x+h)-f(x))\,g(x+h) + f(x)(g(x+h)-g(x))}{h} = \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\cdot g(x+h) + f(x)\cdot\frac{g(x+h)-g(x)}{h}\right)$$

$= f'(x)\cdot g(x) + f(x)\cdot g'(x)$, ty $\lim_{h\to 0} g(x+h) = g(x)$ och

$\lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = f'(x)$; $\lim_{h\to 0} \dfrac{g(x+h)-g(x)}{h} = g'(x)$.