Uppgift 1(a). Med $u_1 = x$, $u_2 = x'$, $u_3 = y$, $u_4 = y'$, har vi

$$\begin{cases} u_1' = u_2 \\ u_2' = -\frac{c}{m} u_2 \sqrt{u_2^2 + u_4^2} \\ u_3' = u_4 \\ u_4' = -g - \frac{c}{m} u_4 \sqrt{u_2^2 + u_4^2} \end{cases}$$

Med vektorbeteckningar har vi standardformen

$$\begin{cases} \boldsymbol{u}' = \boldsymbol{f}(t, \boldsymbol{u}) \\ \boldsymbol{u}(0) = \boldsymbol{u}_0 \end{cases} \quad \text{med } \boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}, \quad \boldsymbol{f}(t, \boldsymbol{u}) = \begin{bmatrix} u_2 \\ -\frac{c}{m} u_2 \sqrt{u_2^2 + u_4^2} \\ u_4 \\ -g - \frac{c}{m} u_4 \sqrt{u_2^2 + u_4^2} \end{bmatrix}, \quad \boldsymbol{u}_0 = \begin{bmatrix} x(0) \\ x'(0) \\ y(0) \\ y'(0) \end{bmatrix}$$

(b).

```
m=0.1; c=0.001; g=9.81; x0=0; vx0=12; y0=1.8; vy0=10; T=3;
f=@(t,u)[u(2)
         -c/m*u(2)*sqrt(u(2)^2+u(4)^2)
         u(4)
         -g-c/m*u(4)*sqrt(u(2)^2+u(4)^2)];
tspan=[0,T]; u0=[x0;vx0;y0;vy0];
[t,U]=ode45(f,tspan,u0);
plot(U(:,1),U(:,3))
```

Uppgift 2(a). Inför $x_i = ih, i = 0, 1, \cdots, n+1$, med $h = \frac{1}{n+1}$.

$$-u''(x_i) + u(x_i) = \lambda u(x_i),\ i = 1, 2, \cdots, n$$

$$-\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + u_i = \lambda u_i,\ i = 1, 2, \cdots, n$$

$$-u_{i-1} + (2 + h^2)u_i - u_{i+1} = h^2 \lambda u_i,\ i = 1, 2, \cdots, n$$

Randvillkoren ger $u_0 = 0$ och $u_{n+1} = 0$.

Matrisformulering med $c = 2 + h^2$ lyder

$$\begin{bmatrix} c & -1 & & & \\ -1 & c & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & c & -1 \\ & & & -1 & c \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n-1} \\ u_n \end{bmatrix} = h^2 \lambda \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n-1} \\ u_n \end{bmatrix}$$

(b). Se laboration 1.

(c).

```
n=30; L=1; k=3;
h=L/(n+1); x=h*[0:n+1]'; e=ones(n,1);
A=spdiags([-e (2+h^2)*e -e],[-1 0 1],n,n);
[V,D]=eigs(A,k,'SM');
V=[zeros(1,k);V;zeros(1,k)]; lambda=diag(D)/h^2;
subplot(3,1,1), plot(x,V(:,1))
subplot(3,1,2), plot(x,V(:,2))
subplot(3,1,3), plot(x,V(:,3))
```

Uppgift 3. Se Lay kapitel 5.

Uppgift 4 (a).

$$\boldsymbol{f}(\boldsymbol{x}) = \begin{bmatrix} x_1^2 + x_2 + (x_1+1)\exp(-x_2) - 5 \\ \exp(x_1 x_2) + x_1 + x_2 - 0.4 \end{bmatrix}$$

$$\boldsymbol{Df}(\boldsymbol{x}) = \begin{bmatrix} 2x_1 + \exp(-x_2) & 1 - (x_1+1)\exp(-x_2) \\ x_2 \exp(x_1 x_2) + 1 & x_1 \exp(x_1 x_2) + 1 \end{bmatrix}$$

(b). Härledning genom linjärisering, se laboration 5.

(c). Newtons metod: $\boldsymbol{x}_{k+1} = \boldsymbol{x}_k + \boldsymbol{d}_k$, där $\boldsymbol{Df}(\boldsymbol{x}_k)\,\boldsymbol{d}_k = -\boldsymbol{f}(\boldsymbol{x}_k)$.

$$\boldsymbol{x}_0 = \begin{bmatrix} -1 \\ 0 \end{bmatrix} : \quad \boldsymbol{Df}(\boldsymbol{x}_0)\,\boldsymbol{d}_0 = -\boldsymbol{f}(\boldsymbol{x}_0) \Leftrightarrow \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} d_{01} \\ d_{02} \end{bmatrix} = - \begin{bmatrix} -4 \\ -0.4 \end{bmatrix} \Leftrightarrow \boldsymbol{d}_0 = \begin{bmatrix} 0.4 \\ 4.4 \end{bmatrix}$$

$$\boldsymbol{x}_1 = \boldsymbol{x}_0 + \boldsymbol{d}_0 = \begin{bmatrix} -0.6 \\ 4.4 \end{bmatrix}$$

(d).

```
f1=@(x1,x2)x1.^2+x2+(x1+1).*exp(-x2)-5;
f2=@(x1,x2)exp(x1.*x2)+x1+x2-0.4;
f=@(x)[f1(x(1),x(2))
       f2(x(1),x(2))];
Df=@(x)[2*x(1)+exp(-x(2)) 1-(x(1)+1)*exp(-x(2))
        x(2)*exp(x(1)*x(2))+1  x(1)*exp(x(1)*x(2))+1];
x=[-1;0];
kmax=10; tol=0.5e-8;
for k=1:kmax
    d=-Df(x)\f(x);
    x=x+d;
    disp([x' norm(d)])
    if norm(d)<tol, break, end
end
```

Uppgift 5. $f(\boldsymbol{x}) = x_1^2 - x_1 + x_1 x_2 + x_2^2$, $\quad \boldsymbol{\nabla} f(\boldsymbol{x}) = \begin{bmatrix} 2x_1 - 1 + x_2 \\ x_1 + 2x_2 \end{bmatrix}$

(a). $\boldsymbol{x}_k = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\quad \boldsymbol{\nabla} f(\boldsymbol{x}_k) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$$\boldsymbol{x}(s) = \boldsymbol{x}_k - s\boldsymbol{\nabla} f(\boldsymbol{x}_k) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} - s \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 - 2s \\ 1 - 3s \end{bmatrix}$$

$$g(s) = f(\boldsymbol{x}(s)) = f(1-2s, 1-3s) = (1-2s)^2 - (1-2s) + (1-2s)(1-3s) + (1-3s)^2 =$$
$$= 2 - 13s + 19s^2$$
$$g'(s) = -13 + 38s = 0 \Leftrightarrow \hat{s} = \tfrac{13}{38}$$
$$\boldsymbol{x}_{k+1} = \boldsymbol{x}_k - \hat{s}\,\boldsymbol{\nabla} f(\boldsymbol{x}_k) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \hat{s}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \tfrac{1}{38}\begin{bmatrix} 12 \\ -1 \end{bmatrix}$$

(b). Vi har $g'(s) = \frac{d}{ds} f(\boldsymbol{x}(s)) = \boldsymbol{\nabla} f(\boldsymbol{x}(s))^T \boldsymbol{x}'(s) = -\boldsymbol{\nabla} f(\boldsymbol{x}(s))^T \boldsymbol{\nabla} f(\boldsymbol{x}_k)$.
Så om $g'(\hat{s}) = 0$ så är $\boldsymbol{\nabla} f(\boldsymbol{x}(\hat{s}))^T \boldsymbol{\nabla} f(\boldsymbol{x}_k) = \boldsymbol{\nabla} f(\boldsymbol{x}_{k+1}) \cdot \boldsymbol{\nabla} f(\boldsymbol{x}_k) = 0$

(c).

```
f=@(x1,x2)x1.^2-x1+x1*x2+x2.^2;
funf=@(x)f(x(1),x(2));
dfdx1=@(x1,x2)2*x1-1+x2; dfdx2=@(x1,x2)x1+2*x2;
gradf=@(x)[dfdx1(x(1),x(2));dfdx2(x(1),x(2))];
kmax=100; smax=10; tol=1e-4;
x=[1;1];
for k=1:kmax;
    grad=gradf(x);
    g=@(s)funf(x-s*grad);
    sk=fminbnd(g,0,smax);
    x=x-sk*grad;
    disp([x' funf(x)])
    if norm(sk*grad)<tol, break, end
end
```

Uppgift 6(a). Inför nät med steglängd h i rumsled och ersätter u''_{xx} med D_+D_- Låter $u_i(t)$ beteckna approximationen av $u(x_i, t)$.

Får följande begynnelsevärdesproblem för ODE-system.

$$\begin{cases} u_i''(t) = \frac{u_{i+1}(t) - 2u_i(t) + u_{i-1}(t)}{h^2}, i = 1, \cdots, n, 0 < t < T \\ u_0(t) = 0, u_{n+1}(t) = \cos(3t) \\ u_i(0) = \sin(\frac{\pi}{2} x_i), i = 1, \cdots, n \\ u_i'(0) = 0, i = 1, \cdots, n \end{cases}$$

På vektorform:

$$\begin{cases} \boldsymbol{U}''(t) = \frac{1}{h^2}(\boldsymbol{b}(t) + \boldsymbol{A}\boldsymbol{U}(t)), 0 < t < T \\ \boldsymbol{U}(0) = \boldsymbol{U}_0,\ \boldsymbol{U}'(0) = \boldsymbol{V}_0 \end{cases}$$

där $\boldsymbol{A}$ är diskreta motsvarigheten till u'' och

$$\boldsymbol{U}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_{n-1}(t) \\ u_n(t) \end{bmatrix} \quad \boldsymbol{b}(t) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \cos(3t) \end{bmatrix} \quad \boldsymbol{U}_0 = \begin{bmatrix} \sin(\frac{\pi}{2}x_1) \\ \sin(\frac{\pi}{2}x_2) \\ \vdots \\ \sin(\frac{\pi}{2}x_{n-1}) \\ \sin(\frac{\pi}{2}x_n) \end{bmatrix} \quad \boldsymbol{V}_0 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

Slutligen får vi formulera om vår ODE som ett första ordningens system.

$$\begin{cases} \boldsymbol{W}'(t) = \boldsymbol{f}(t, \boldsymbol{W}), 0 < t < T \\ \boldsymbol{W}(0) = \boldsymbol{W}_0 \end{cases}$$

där

$$\boldsymbol{W} = \begin{bmatrix} \boldsymbol{U} \\ \boldsymbol{V} \end{bmatrix}, \quad \boldsymbol{f}(t, \boldsymbol{W}) = \begin{bmatrix} \boldsymbol{V}(t) \\ \frac{1}{h^2}(\boldsymbol{b}(t) + \boldsymbol{A}\boldsymbol{U}(t)) \end{bmatrix}, \quad \boldsymbol{W}_0 = \begin{bmatrix} \boldsymbol{U}_0 \\ \boldsymbol{V}_0 \end{bmatrix}$$

(b).

```
T=5; L=1; u0=@(x)sin(pi/2*x);
n=30; h=L/(n+1); xi=h*[1:n]'; tspan=linspace(0,T,n+1);
U0=u0(xi); V0=zeros(size(xi)); W0=[U0; V0];
A=spdiags(ones(n,1)*[-1 2 -1],[-1 0 1],n,n);
b=@(t)[0;zeros(n-2,1);cos(3*t)];
hyperbol=@(t,w)[w(n+1:2*n);1/h^2*(b(t)-A*w(1:n))];
[t,W]=ode45(hyperbol,tspan,W0);
x=[0;xi;L]; U=[zeros(size(t)),W(:,1:n),cos(3*t)];  % Kantar med randvärden
surf(x,t,U)
xlabel('x'), ylabel('t')
```