Uppgift 1. Se laboration 1.

Uppgift 2(a). För olika x_i-värden skall vi lösa $f(w) = x - w\exp(w) = 0$ för att få värdet på $W(x_i)$. Då $x = 0$ gäller att $w = 0$.

```
f=@(w,x)x-w*exp(w);
x=linspace(0,20); W=zeros(size(x));
W(1)=0;
for i=2:length(x)
    W(i)=fzero(@(w)f(w,x(i)),W(i-1));
end
plot(x,W,'r')
```

(b). Vi har $1 = W'(x)(1+W(x))\exp(W(x))$, dvs. $W'(x) = \frac{1}{1+W(x)}\exp(-W(x))$ samt $W(0) = 0$.

```
f=@(x,w)1/(1+w)*exp(-w); W0=0;
[x,W]=ode45(f,x,W0);
plot(x,W,'b')
```

Uppgift 3(a). Inför $x_i = ih, i = 0, 1, \cdots, n+1$, med $h = \frac{1}{n+1}$.

$$u''(x_i) + \cos(x_i)u(x_i) = \exp(x_i),\ i = 1, 2, \cdots, n$$

$$\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + \cos(x_i)u_i = \exp(x_i),\ i = 1, 2, \cdots, n$$

$$u_{i-1} + (-2 + h^2\cos(x_i))u_i + u_{i+1} = h^2\exp(x_i),\ i = 1, 2, \cdots, n$$

Randvillkoret $u'(0) = 1$ approximeras av $\frac{u_1 - u_0}{h} = 1$, dvs. $u_0 = u_1 - h$, och randvillkoret $u(1) = 3$ ger $u_{n+1} = 3$.

Matrisformulering med $c_1 = -1 + h^2\cos(x_1),\ c_i = -2 + h^2\cos(x_i),\ i = 2, \cdots, n$, lyder

$$\begin{bmatrix} c_1 & 1 & & & \\ 1 & c_2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & c_{n-1} & 1 \\ & & & 1 & c_n \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n-1} \\ u_n \end{bmatrix} = \begin{bmatrix} h^2\exp(x_1) + h \\ h^2\exp(x_2) \\ \vdots \\ h^2\exp(x_{n-1}) \\ h^2\exp(x_n) - 3 \end{bmatrix}$$

(b). Se laboration 1.

(c).

```
n=30; L=1;
h=L/(n+1); xi=h*[1:n]'; e=ones(n,1);
A=spdiags([-e (2-h^2*cos(xi)) -e],[-1 0 1],n,n); A(1,1)=A(1,1)-1;
b=-h^2*exp(xi); b(1)=b(1)-h; b(n)=b(n)+3;
u=A\b;
x=[0;xi;L]; u=[u(1)-h;u;3];
plot(x,u)
```

Uppgift 4(a). Egenvärdesproblemet:

$$\det(\boldsymbol{A}-\lambda\boldsymbol{I}) = \left|\begin{bmatrix} 4-\lambda & -5 \\ -2 & 1-\lambda \end{bmatrix}\right| = (4-\lambda)(1-\lambda)-10 =$$

$$= \lambda^2 - 5\lambda - 6 = (\lambda-6)(\lambda+1) = 0$$

Egenvektorn som hör ihop med $\lambda_1 = -1$:

$$(\boldsymbol{A}-\lambda_1\boldsymbol{I})\boldsymbol{v} = (\boldsymbol{A}+\boldsymbol{I})\boldsymbol{v} = \begin{bmatrix} 5 & -5 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ ger } \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Egenvektorn som hör ihop med $\lambda_2 = 6$:

$$(\boldsymbol{A}-\lambda_2\boldsymbol{I})\boldsymbol{v} = (\boldsymbol{A}-6\boldsymbol{I})\boldsymbol{v} = \begin{bmatrix} -2 & -5 \\ -2 & -5 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ ger } \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$$

Lösningen till ODE-systemet:

$$\boldsymbol{u}(t) = c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix}\exp(-t) + c_2\begin{bmatrix} 5 \\ -2 \end{bmatrix}\exp(6t)$$

$$\boldsymbol{u}(0) = c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 5 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \end{bmatrix} \Rightarrow \boldsymbol{c} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$$\boldsymbol{u}(t) = 3\begin{bmatrix} 1 \\ 1 \end{bmatrix}\exp(-t) - \begin{bmatrix} 5 \\ -2 \end{bmatrix}\exp(6t)$$

(b). Se laboration 4 eller Lay kapitel 5.

(c). Euler framåt: $\boldsymbol{u}_{n+1} = \boldsymbol{u}_n + h\boldsymbol{A}\boldsymbol{u}_n = (\boldsymbol{I}+h\boldsymbol{A})\boldsymbol{u}_n, n = 0, 1, \cdots, \boldsymbol{u}_0 = [\,-2\ \ 5\,]^T$

$$\boldsymbol{u}_1 = (\boldsymbol{I}+0.1\boldsymbol{A})\boldsymbol{u}_0 = \begin{bmatrix} 1.4 & -0.5 \\ -0.2 & 1.1 \end{bmatrix}\begin{bmatrix} -2 \\ 5 \end{bmatrix} = \begin{bmatrix} -5.3 \\ 5.9 \end{bmatrix}$$

Uppgift 5(a).

$$\boldsymbol{f}(\boldsymbol{x}) = \begin{bmatrix} x_1 + x_2^3 - 1 \\ x_1\exp(x_1x_2) + x_2 + 1 \end{bmatrix}, \quad \boldsymbol{Df}(\boldsymbol{x}) = \begin{bmatrix} 1 & 3x_2^2 \\ (1+x_1x_2)\exp(x_1x_2) & x_1^2\exp(x_1x_2)+1 \end{bmatrix}$$

(b). Se laboration 5.

(c). Newtons metod: $\boldsymbol{x}_{k+1} = \boldsymbol{x}_k + \boldsymbol{d}_k$, där $\boldsymbol{Df}(\boldsymbol{x}_k)\,\boldsymbol{d}_k = -\boldsymbol{f}(\boldsymbol{x}_k)$

$$\boldsymbol{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}: \quad \boldsymbol{Df}(\boldsymbol{x}_0)\,\boldsymbol{d}_0 = -\boldsymbol{f}(\boldsymbol{x}_0) \Leftrightarrow \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} d_{01} \\ d_{02} \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix} \Leftrightarrow \boldsymbol{d}_0 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$\boldsymbol{x}_1 = \boldsymbol{x}_0 + \boldsymbol{d}_0 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

(d).

```
f1=@(x1,x2)x1+x2.^3-1; f2=@(x1,x2)x1.*exp(x1.*x2)+x2+1;
f=@(x)[f1(x(1),x(2));f2(x(1),x(2))];
Df=@(x)[1 3*x(2)^2; (1+x(1)*x(2))*exp(x(1)*x(2)) x(1)^2*exp(x(1)*x(2))+1];
x=[1;0];
kmax=10; tol=0.5e-8;
for k=1:kmax
    d=-Df(x)\f(x);
    x=x+d;
    disp([x' norm(d)])
    if norm(d)<tol, break, end
end
```

Uppgift 6(a). Inför nät med steglängden h i rumsled och ersätt u''_{xx} med D_+D_- och u'_x med D_-. Låt $u_i(t)$ beteckna approximationen av $u(x_i,t)$. För differentialekvationen får vi

$$u'_i(t) = \frac{u_{i+1}(t) - 2u_i(t) + u_{i-1}(t)}{h^2} + f(x_i,t), \quad i = 1,\cdots,n, \quad 0 < t < T$$

och randvillkoren ger

$$u_0(t) = g(t), \quad u_{n+1}(t) = u_n(t), \quad 0 \le t \le T$$

Begynnelsevillkoren blir

$$u_i(0) = 0, \quad i = 1,\cdots,n$$

Begynnelsevärdesproblemet för ODE blir

$$\begin{cases} \boldsymbol{U}'(t) = \frac{1}{h^2}(\boldsymbol{b}(t) - \boldsymbol{A}\boldsymbol{U}(t)) + \boldsymbol{c}(t), & 0 < t < T \\ \boldsymbol{U}(0) = \boldsymbol{U}_0 \end{cases}$$

där

$$\boldsymbol{b}(t) = \begin{bmatrix} g(t) \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \quad \boldsymbol{A} = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 1 \end{bmatrix} \quad \boldsymbol{c}(t) = \begin{bmatrix} f(x_1,t) \\ f(x_2,t) \\ \vdots \\ f(x_{n-1},t) \\ f(x_n,t) \end{bmatrix}$$

och

$$\boldsymbol{U}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_{n-1}(t) \\ u_n(t) \end{bmatrix} \quad \boldsymbol{U}_0 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

(b).

```
n=30; L=5; T=25; g=@(t)1+sin(0.7*t);
h=L/(n+1); xi=h*(1:n)';
A=spdiags(ones(n,1)*[-1 2 -1],[-1 0 1],n,n); A(n,n)=1;
b=@(t)[g(t);zeros(n-2,1);0]; c=@(t)0.2*xi*exp(-0.2*t);
f=@(t,u)(b(t)-A*u)/h^2+c(t);
tspan=linspace(0,T,n+1); U0=zeros(n,1);
[t,U]=ode45(f,tspan,U0);
x=[0;xi;L]; U=[g(t),U,U(:,n)];
subplot(1,2,1)
surf(x,t,U), xlabel('x'), ylabel('t')
subplot(1,2,2)
contourf(x,t,U,20), xlabel('x'), ylabel('t')
```