1. a)

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{1/x} \overset{H}{=} \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+}(-x) = 0$$

↑

Lösningar övningstenta 2

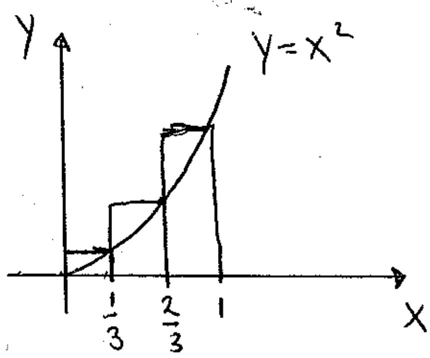
↓

b) Lös ut x ur ekvationen $y = f(x)$. Vi får då $x = 2 \pm \sqrt{4+y}$. Eftersom $x \leq 2$ måste vi välja $x = 2 - \sqrt{4+y}$. Om vi presenterar inversen som funktion av ~~x~~ får vi $f^{-1}(x) = 2 - \sqrt{4+x}$; $D_{f^{-1}} = [-4, \infty)$

c)

$$R_3 = f\left(\tfrac{1}{3}\right)\cdot\tfrac{1}{3} + f\left(\tfrac{2}{3}\right)\cdot\tfrac{1}{3} + f\left(\tfrac{3}{3}\right)\cdot\tfrac{1}{3} =$$

$$= \frac{1}{27}(1+4+9) = \frac{14}{27}.$$



d)

$$\int \frac{e^x}{e^{2x}+e^x-2}\,dx = \left[\begin{array}{l} e^x = t \\ dt = e^x dx \end{array}\right] = \int \frac{1}{t^2+t-2}\,dt =$$

$$= \int \frac{1}{(t-1)(t+2)}\,dt = \int\left(\frac{A}{t-1} + \frac{B}{t+2}\right)dt = \int\left(\frac{1/3}{t-1} + \frac{-1/3}{t+2}\right)dt$$

$$= \frac{1}{3}\ln|t-1| - \frac{1}{3}\ln|t+2| + C$$

$$= \frac{1}{3}\ln\left|\frac{e^x-1}{e^x+2}\right| + C.$$

2\. a) Se föreläsningsanteckningar (eller boken)

b) $\int x^2 \sin(x^3)\,dx = \left\{ \begin{array}{l} t = x^3 \\ dt = 3x^2\,dx \end{array} \right\} =$

$= \frac{1}{3}\int \sin t\,dt = -\frac{1}{3}\cos t + C = -\frac{1}{3}\cos(x^3) + C$

3\. a) Se föreläsningsanteckningar (eller boken)

b) medelvärdessatsen

$$f'(c) = \frac{f(4) - f(2)}{4-2} = \frac{3-1}{2} = 1$$

men $f'(x) > 1$ för alla x, finns ingen funktion f som uppfyller $f'(x) > 1$ för alla x.

4\. $f(x) = \frac{1}{x} + \frac{1}{2}\ln x + \arctan x$ är definerad för alla $x > 0$. Dess derivata är

$$f'(x) = -\frac{1}{x^2} + \frac{1}{2x} + \frac{1}{x^2+1} = \frac{x^3 + x - 2}{2x^2(x^2+1)} = \frac{(x-1)\left(\left(x+\frac{1}{2}\right)^2 + \frac{7}{4}\right)}{2x^2(x^2+1)}$$

$f(x) \to \infty$ då $x \to 0^+$

$f(x) \to \infty$ då $x \to \infty$

x		1	
f'	$-$	0	$+$
f	$\searrow$	$1+\frac{\pi}{4}$	$\nearrow$

lok. min.

$\Rightarrow$

$D_f = (0, \infty)$, $V_f = \left(1 + \frac{\pi}{4}, \infty\right)$

5. $\displaystyle\int_2^\infty \frac{x-8}{x^3+4x}\,dx$

lsg $\displaystyle\frac{x-8}{x^3+4x} = \frac{x-8}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4} =$

$$= \frac{A(x^2+4)+(Bx+C)x}{x(x^2+4)} = \frac{(A+B)x^2+Cx+4A}{x(x^2+4)}$$

$$\begin{cases} A+B = 0 \\ C = 1 \\ 4A = -8 \end{cases} \Leftrightarrow \begin{array}{l} B = 2 \\ C = 1 \\ A = -2 \end{array}$$

$$\Rightarrow \int_2^\infty \frac{x-8}{x^3+4x}\,dx = \lim_{t\to\infty}\int_2^t \left(-\frac{2}{x} + \frac{2x+1}{x^2+4}\right)dx$$

$$= \lim_{t\to\infty}\int_2^t \left(\frac{2x}{x^2+4} + \frac{1}{x^2+4} - \frac{2}{x}\right)dx$$

$$= \lim_{t\to\infty}\left[\Big[\ln(x^2+4)\Big]_2^t + \Big[\tfrac{1}{2}\arctan\tfrac{x}{2}\Big]_2^t - \Big[2\ln x\Big]_2^t\right] =$$

$$= \lim_{t\to\infty}\Big[\ln(t^2+4) - \overset{\ln 2^3 = 3\ln 2}{\ln 8} + \tfrac{1}{2}\arctan\tfrac{t}{2} - \tfrac{1}{2}\arctan 1 - 2\ln t + 2\ln 2\Big]$$

$$= \lim_{t\to\infty}\Big[\underbrace{\ln\frac{t^2+4}{t^2}}_{\to 0} + \underbrace{\tfrac{1}{2}\arctan\tfrac{t}{2}}_{\to \pi/2} - \frac{\pi}{8} - \ln 2\Big] = \frac{\pi}{8} - \ln 2$$

6. a) $x^2 y' = y + 1$

$$y' - \frac{1}{x^2} y = \frac{1}{x^2}$$

multiplicera båda leden med integrerande faktorn

$$e^{\int -1/x^2 dx} = e^{1/x}$$

och vi får

$$\frac{d}{dx}\left(e^{1/x} y\right) = \frac{1}{x^2} e^{1/x}$$

$$e^{1/x} y = -e^{-1/x} + C \iff \underline{\underline{y = Ce^{-1/x} - 1}}$$

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b) $x^2 y' = y + 1$

$$\frac{1}{y+1} dy = \frac{1}{x^2} dx$$

$$\int \frac{1}{y+1} dy = \int \frac{1}{x^2} dx$$

$$\ln|y+1| = -\frac{1}{x} + D$$

$$|y+1| = e^{-1/x + D}$$

$$y+1 = \underbrace{\pm e^{D}}_{C} e^{-1/x}$$

$$y = Ce^{-1/x} - 1$$

7. a) Falskt,

$f(x) = \arctan x$ är strängt växande, men $f(x)$ går inte mot ∞, när $x \to \infty$.

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b) Falskt,

tag t.ex. $f(x) = 1$ på $[0,2]$

$$\int_0^2 \sqrt{1}\, dx = 2 \neq \sqrt{2} = \sqrt{\int_0^2 1\, dx}$$

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c) Sant,

med $f(x) = \dfrac{1}{1+x^2}$ och $x_i^* = x_i = a + i\Delta x = \dfrac{i}{n}$

så är

$$\sum_{i=1}^{n} \frac{n}{n^2+i^2} = \sum_{i=1}^{n} \frac{1}{1+\left(\frac{i}{n}\right)^2} \cdot \frac{1}{n} = \sum_{i=1}^{n} f(x_i^*)\,\Delta x.$$

8. $$\int_0^1 \frac{1}{\sqrt{1+x^2}}\,dx = \left[\begin{array}{l|l} x = \tan\theta & 0 \le \tan\theta \le 1 \;; \\ dx = \dfrac{1}{\cos^2\theta}\,d\theta & 0 \le \theta \le \pi/4 \\ & \dfrac{1}{\sqrt{1+\tan^2\theta}} = |\cos\theta| = \cos\theta \end{array}\right]$$

$$= \int_0^{\pi/4} \frac{\cos\theta}{\cos^2\theta}\,d\theta = \int_0^{\pi/4} \frac{\cos\theta}{1-\sin^2\theta}\,d\theta = \left[\begin{array}{l} \sin\theta = t \\ dt = \cos\theta\,d\theta \\ \theta = 0 \Leftrightarrow t = 0 \;;\; \theta = \frac{\pi}{4} \Leftrightarrow t = \frac{1}{\sqrt{2}} \end{array}\right]$$

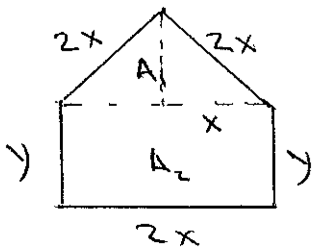
$$= \int_0^{1/\sqrt{2}} \frac{1}{1-t^2}\,dt = \int_0^{1/\sqrt{2}} \left(\frac{A}{1-t} + \frac{B}{1+t}\right) dt = \left[\begin{array}{l} A = 1/2 \\ B = 1/2 \end{array}\right] =$$

$$= \int_0^{1/\sqrt{2}} \left(\frac{1/2}{1-t} + \frac{1/2}{1+t}\right) dt = \frac{1}{2}\Big[-\ln|1-t| + \ln|1+t|\Big]_0^{1/\sqrt{2}}$$

$$= \frac{1}{2}\left[\ln\left(\frac{1+t}{1-t}\right)\right]_0^{1/\sqrt{2}} = \frac{1}{2}\ln\left(\frac{1+1/\sqrt{2}}{1-1/\sqrt{2}}\right) =$$

$$= \frac{1}{2}\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right).$$

9. Beteckningar enligt figur!



$$A_1(x) = \frac{2x \cdot \sqrt{3}x}{2} = \sqrt{3}x^2$$

$$A_2(x) = 2x \cdot y$$

$$\Rightarrow 2xy + \sqrt{3}x^2 = A \quad \Rightarrow y = \frac{A - \sqrt{3}x^2}{2x}$$

Omkretsen blir då

$$O(x) = 6x + 2 \cdot \frac{A - \sqrt{3}x^2}{2x} = 6x + \frac{A - \sqrt{3}x^2}{x} =$$

$$= (6 - \sqrt{3})x + \frac{A}{x} \quad ; \quad 0 < x < \sqrt{\frac{A}{\sqrt{3}}}$$

derivera $O(x)$

$$O'(x) = 6 - \sqrt{3} - \frac{A}{x^2} = 0 \quad \text{då} \quad x = \sqrt{\frac{A}{6-\sqrt{3}}}$$

x		$\sqrt{\frac{A}{6-\sqrt{3}}}$	
$O'(x)$	$-$	0	$+$
$O(x)$	↘		↗
		lok. min	

$$O\left(\sqrt{\frac{A}{6-\sqrt{3}}}\right) = (6-\sqrt{3}) \cdot \sqrt{\frac{A}{6-\sqrt{3}}} + \frac{A}{\sqrt{\frac{A}{6-\sqrt{3}}}} =$$

$$= \frac{(6-\sqrt{3})\frac{A}{(6-\sqrt{3})} + A}{\frac{\sqrt{A}}{\sqrt{6-\sqrt{3}}}} = 2\sqrt{A}\sqrt{6-\sqrt{3}}$$