1 a) $\int \frac{3}{2x^2+4}\,dx = \frac{3}{2}\int \frac{1}{x^2+(\sqrt{2})^2}\,dx = \frac{3}{2\sqrt{2}}\arctan\frac{x}{\sqrt{2}} + c$

---

b) $\int \frac{x}{x^2+5x+6}\,dx = \int\left(\frac{A}{x+3} + \frac{B}{x+2}\right)dx =$

$= \int\left(\frac{3}{x+3} - \frac{2}{x+2}\right)dx = 3\ln|x+3| - 2\ln|x+2| + c$

---

c) $\int \cos^3 x\,dx = \int \cos^2 x \cos x\,dx = \int (1-\sin^2 x)\cos x\,dx$

$= \left[\begin{array}{l} \sin x = t \\ dt = \cos x\,dx \end{array}\right] = \int (1-t^2)\,dt = t - \frac{t^3}{3} + c =$

$= \sin x - \frac{\sin^3 x}{3} + c.$

---

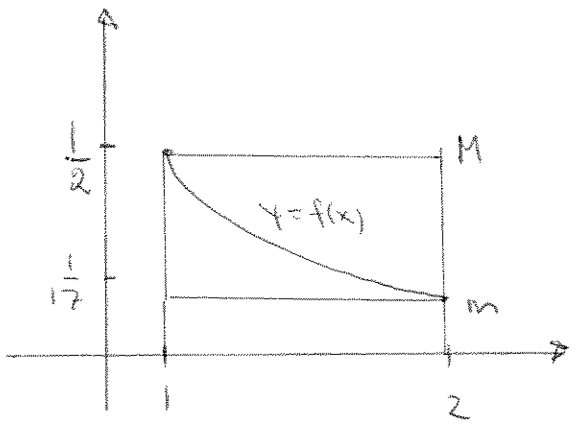
d) $$\frac{|x|\sqrt{12 - 4/x^2}}{x(1 - 4/x)} = \{x<0\} = \frac{-x\sqrt{12 - 4/x^2}}{x(1 - 4/x)} =$$

$$= -\frac{\sqrt{12 - 4/x^2}}{1 - 4/x} \to -\sqrt{12} = -2\sqrt{3}, \text{ då } x \to -\infty$$

---

e)

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$$



$\Rightarrow$

$$\frac{1}{17}(2-1) \le \int_1^2 \frac{1}{1+x^4}\,dx \le \frac{1}{2}\cdot(2-1)$$

$$\frac{1}{17} \le \int_1^2 \frac{1}{1+x^4}\,dx \le \frac{1}{2}.$$

2. a) $$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$

om gränsvärdet existerar.

b) $$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \int_0^1 \frac{1}{x^{1/2}}\,dx$$ konvergent $\alpha = \frac{1}{2} < 1$

$$\int_0^{12} \frac{1}{x-2}\,dx = \cancel{\lim_{t\to 2^+} \int_t^{12} \frac{1}{x-2}\,dx + \int_0}$$

$$\cancel{= \lim_{t\to 0^+} \ln|x-2| \Big|_t^{12} = \ln 10 - \lim_{t\to 0^+} \ln|x}$$

$$= \int_0^2 \frac{1}{x-2}\,dx + \int_2^{12} \frac{1}{x-2}\,dx = \int_0^{12} \frac{1}{x-2}\,dx$$

om båda integralerna i VL existerar,

men det gör de ej!

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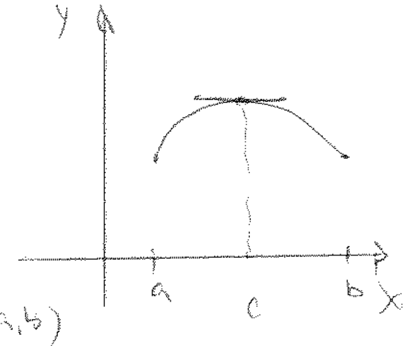
3. 1. f kont på $[a,b]$

2. f deriverbar på (a,b)

3. $f(a) = f(b)$



då finns minst ett tal $c \in (a,b)$

så $f'(c) = 0$

3.b) $f(-1) = 0$,

$$f'(x) = \frac{1}{2\sqrt{x^3+1}} \cdot 3x^2 = 0 \quad \text{då} \quad x = 0.$$

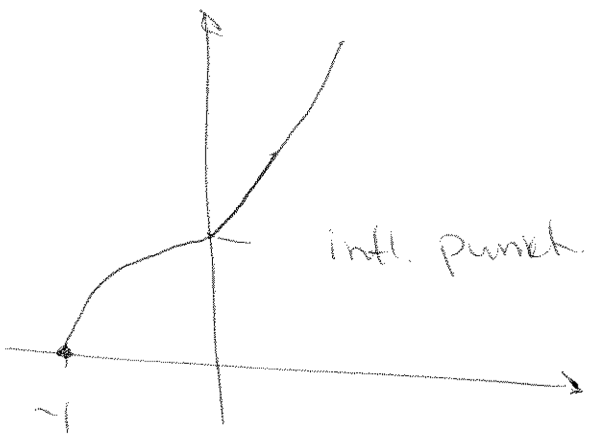
$$f''(x) = \frac{6x \cdot 2\sqrt{x^3+1} - 3x^2 \cdot \frac{1}{\sqrt{x^3+1}} \cdot 3x^2}{4(x^3+1)} =$$

$$= \frac{12x(x^3+1) - 9x^4}{4(x^3+1)\sqrt{x^3+1}} = \frac{12x^4 + 12x - 9x^4}{4(x^3+1)\sqrt{x^3+1}}$$

$$= \frac{3x^4 + 12x}{N} = 3x(\underbrace{x^3+4}_{>0})$$

$\rightarrow$ $f''(x) < 0$ då $-1 < x < 0$ Konkav

$f''(x) > 0$ då $x > 0$ Konvex



4. a) $\frac{f(x)}{x} = \frac{x^2-3}{x^2+2x} \to 1$ då $x \to \pm\infty$

$f(x) - kx = \frac{x^2-3}{x+2} - x = \frac{\cancel{x^2} - 3 - \cancel{x^2} - 2x}{x+2}$

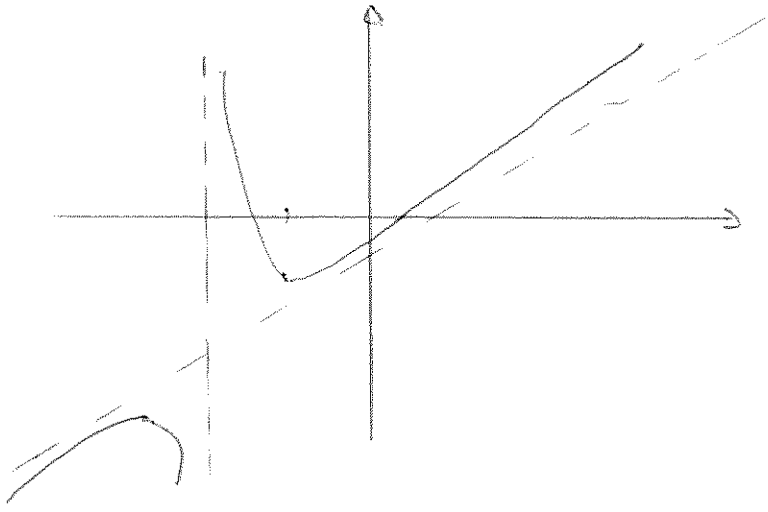
$\to -2$ då $x \to \pm\infty$

$\boxed{y = x-2}$ $\boxed{x = -2 \text{ lodrät}}$ VISA!

$$f'(x) = \frac{2x(x+2) - (x^2-3)\cdot 1}{(x+2)^2} =$$

$$= \frac{2x^2+4x-x^2+3}{(x+2)^2} = \frac{x^2+4x+3}{(x+2)^2} = \frac{(x+1)(x+3)}{(x+2)^2}$$

x	$-\infty$		-3		-2		-1		∞
f'		$+$	0	$-$		$-$	0	$+$	
f	$-\infty$		-6		$-\infty$ / $+\infty$		-2		∞



5. $$\int_1^{\infty} \frac{2}{x^3+x}\,dx = \int_1^{\infty} \frac{2}{x(x^2+1)}\,dx =$$

$$= \int_1^{\infty} \left(\frac{A}{x} + \frac{Bx+C}{x^2+1} \right) dx = \left\{ \begin{matrix} A=2 & C=0 \\ B=-2 & \end{matrix} \right\} =$$

$$= \int_1^{\infty} \left(\frac{2}{x} - \frac{2x}{x^2+1} \right) dx =$$

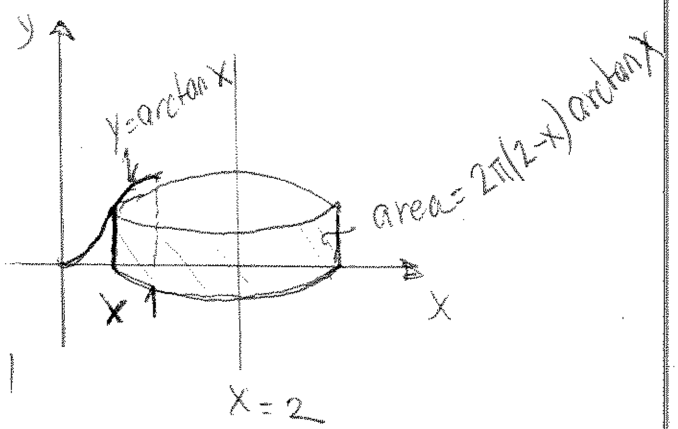
$$\lim_{u \to \infty} \Big[2\ln x - \ln(x^2+1) \Big]_1^u =$$

$$= \lim_{u \to \infty} \ln \frac{u^2}{u^2+1} + \ln 2 =$$

$$= \ln\left(\lim_{u \to \infty} \frac{u^2}{u^2+1} \right) + \ln 2 = \ln 1 + \ln 2 = \underline{\ln 2}$$

6. Beräkna volymen av den kropp som uppstår då området givet av $0 \le y \le \arctan x$ och $0 \le x \le 1$ roteras ett varv kring linjen $x = 2$.

Lösn Volymen ges av



$$V = \int_0^1 2\pi(2-x)\arctan x \, dx =$$

$$= \{PI\} = 2\pi\left(\left[\left(2x - \frac{x^2}{2}\right)\arctan x\right]_0^1 - \int_0^1 \left(2x - \frac{x^2}{2}\right) \cdot \frac{1}{1+x^2}\, dx\right) =$$

$$= 2\pi\left(\frac{3\pi}{8} - \int_0^1 \left(\frac{2x}{1+x^2} - \frac{x^2/2}{1+x^2}\right) dx\right)$$

$$= 2\pi\left(\frac{3\pi}{8} - \int_0^1 \left(\frac{2x}{1+x^2} + \frac{1/2}{1+x^2} - \frac{1}{2}\,\frac{1+x^2}{1+x^2}\right) dx\right)$$

$$= 2\pi\left(\frac{3\pi}{8} - \left[\ln(1+x^2) + \frac{1}{2}\arctan x - \frac{x}{2}\right]_0^1\right) =$$

$$= 2\pi\left(\frac{3\pi}{8} - \left(\ln 2 + \frac{\pi}{8} - \frac{1}{2}\right)\right) =$$

$$= 2\pi\left(\frac{\pi}{4} + \frac{1}{2} - \ln 2\right).$$

4.

a) $$\int_{-5}^{5} (x^7 + \sin x + x + 2)\,dx =$$

$$= \underbrace{\int_{-5}^{5} (\underbrace{x^7 + \sin x + x}_{\text{udda}})\,dx}_{=0} + \int_{-5}^{5} 2\,dx = 2x\Big|_{-5}^{5} = 20$$

Sant.

---

b) ~~Sant.~~

~~$f(x) = |x|x$ har en inflexionspunkt i $x=0$,~~

~~men $f''(0)$ existerar inte.~~

---

c) Falskt!

$$\frac{1}{\sqrt{1+x^2}} \overset{x \geq 1}{\geq} \frac{1}{\sqrt{x^2+x^2}} = \frac{1}{\sqrt{2}\,x}$$

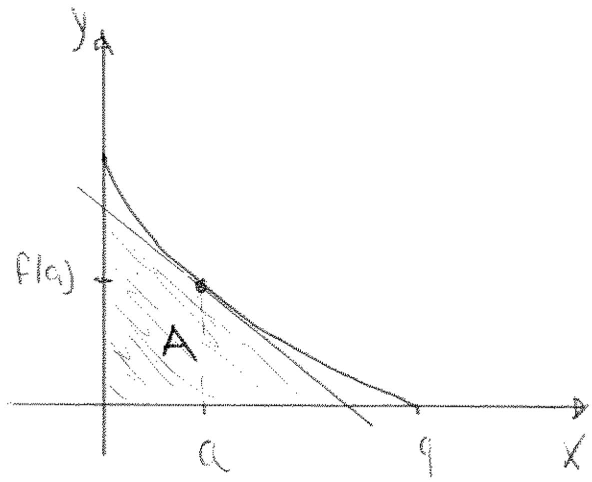
$$\int_{1}^{\infty} \frac{1}{\sqrt{2}\,x}\,dx \text{ är divergent} \Rightarrow$$

$$\Rightarrow \int_{1}^{\infty} \frac{1}{\sqrt{1+x^2}}\,dx \text{ är divergent}$$

8. <u>Tang. ekv.</u>

$$y - f(a) = f'(a)(x-a)$$

$$\Rightarrow \quad y - (3-\sqrt{a}) = -\frac{1}{2\sqrt{a}}(x-a)$$



$x = 0$ ger $y = \dfrac{6-\sqrt{a}}{2}$

$y = 0$ ger $x = 6\sqrt{a} - a$

<u>Areafunktionen blir då</u>

$$A(a) = \frac{(6-\sqrt{a})(6\sqrt{a}-a)}{4} = \frac{36\sqrt{a} - 12 + a\sqrt{a}}{4};$$

<u>derivera!</u>

$$A'(a) = \frac{\frac{36}{2\sqrt{a}} - 12 + \frac{3}{2}\sqrt{a}}{4} = \frac{36 - 24\sqrt{a} + 3a}{8\sqrt{a}}$$

<u>Lös $A'(a) = 0$</u>

$$36 + 3a = 24\sqrt{a} \iff a + 12 = 8\sqrt{a} \Rightarrow (a+12)^2 = 64a$$

$$\iff a^2 - 40a + 144 = 0 \iff a_1 = 4 \quad \text{ok!}$$
$$a_2 = 36 \quad \text{(falsk!)}$$

<u>Vi har då</u>

$A(0) = 0$ (avläses i figuren!) $\quad A(4) = 8 \quad A(9) = \dfrac{27}{4}$

Största värde är då 8.

9. Torricellis lag

$$\frac{dV}{dt} = k\sqrt{h(t)}$$, där $V(t) = \pi R^2 h(t)$

k = Proportionalitetskonstant.

$h(t)$ = höjden av vattnet i tanken vid tiden t timmar.

$$\Rightarrow \frac{d}{dt}\left(\pi R^2 h(t)\right) = k\sqrt{h(t)} \Rightarrow$$

$$\frac{1}{\sqrt{h}} \cdot \frac{dh}{dt} = P$$, där $P = \frac{k}{\pi R^2}$

Som är en separabel differentialekvation med lösningen

$$\int \frac{1}{\sqrt{h}}\, dh = \int P\, dt$$

Som ger oss

$$2\sqrt{h(t)} = Pt + c$$

Från texten vet vi att $h(0) = 5$ om att $h(1) = 2$.

Insättnin av $h(0) = 5$ ger

$$2\sqrt{h(0)} = P \cdot 0 + c \Leftrightarrow 2\sqrt{5} = c$$

Konstanten P bestämmer vi genom $h(1) = 2$ och får att $P = 2(\sqrt{2} - \sqrt{5})$.

$$\Rightarrow \sqrt{h} = (\sqrt{2} - \sqrt{5})t + \sqrt{5}$$

Vi söker tömningstiden T, d.v.s. $h(T) = 0$

$$\Rightarrow 0 = (\sqrt{2} - \sqrt{5})T + \sqrt{5} \Leftrightarrow T = \frac{\sqrt{5}}{\sqrt{5} - \sqrt{2}}.$$