1. Solutions to exercises

1.1. Chapter 1

1. (a) $7-i$.
(b) $3+3$i.
(c) $-11-2i$.
(d) 5.
(e) $-2+3i$.
2. (a) $\mathrm{Re}(\frac{z-a}{z+a})=\frac{|z|^2-a^2}{|z|^2+a^2+2a\mathrm{Re}(z)}$, $\mathrm{Im}(\frac{z-a}{z+a})=\frac{2a\mathrm{Im}(z)}{|z|^2+a^2+2a\mathrm{Re}(z)}$.
(b) $\frac{19}{25}-\frac{8}{25}i$.
(c) 1.
(d) 1 if $n=4k$, $k\in\mathbb{Z}$, i if $n=4k+1$, $k\in\mathbb{Z}$, -1 if $n=4k+2$,$k\in\mathbb{Z}$, $-i$ if $n=4k+3$,$k\in\mathbb{Z}$.
3. (a) $\sqrt{5}$, $-2-i$.
(b) $5\sqrt{5}$, $5-10i$.
(c) $\sqrt{\frac{10}{11}}$, $\frac{3}{11}(\sqrt{2}-1)+\frac{i}{11}(\sqrt{2}+9)$.
(d) 8, $8i$.
4.(a) $2e^{i\frac{\pi}{2}}$.
(b) $\sqrt{2}e^{i\frac{\pi}{4}}$.
(c) $2\sqrt{3}e^{i\frac{5\pi}{6}}$.
(d) $e^{i\frac{3\pi}{2}}$.
(e) $\sqrt{13}e^{-i\tan^{-1}(\frac{2}{3})}$.
(f) 5.
(g) $\sqrt{6}e^{-i\tan^{-1}(\frac{1}{\sqrt{5}})}$.
(h) $\frac{4}{9}e^{i\pi}$.
5.(a) $-1+i$.
(b) $34i$.
(c) -1.
(d) 2.
6. (a) $\cos(\log(2))+i\sin(\log(2))=e^{i\log(2)}$.
(b) $5i=5e^{i\frac{\pi}{2}}$.
(c) $ei=e\cdot e^{i\frac{\pi}{2}}$.
(d) $e^{\phi}\sqrt{2}\cos(\frac{\pi}{4}+\phi)+ie^{\phi}\sqrt{2}\sin(\frac{\pi}{4}+\phi)=e^{\phi}\sqrt{2}e^{i(\frac{\pi}{4}+\phi)}$.
8.(a) $z=\pm5$.
(b) $z=\frac{1}{2}(-1\pm3i)$.
(c) $z=\frac{1}{5}(-2\pm i)$.
(d) $z=\frac{1}{2}(1\pm\sqrt{3})$.
(e) $z=0,2$.
10. (a) $z=e^{i\frac{k\pi}{3}}$, $k=0,1,...,5$.
(b) $z=2e^{i\frac{(2k+1)\pi}{4}}$, $k=0,1,2,3$.
(c) $z=\sqrt[3]{3}e^{i\frac{(2k+1)\pi}{6}}$, $k=0,1,...,5$.
(d) $z=e^{i\frac{2k\pi}{3}}$, $k=0,1,2,3$ and $z=\sqrt[3]{2}e^{i\frac{2k\pi}{3}}$, $k=0,1,2,3$.
13. $z=e^{i\frac{\pi}{4}}-1$ and $z=e^{i\frac{5\pi}{4}}-1$.
24. (a) open, bounded and connected.

(b) open, not bounded, connected.
(c) open, bounded and connected.
(d) closed, bounded and connected.
(e) open, bounded and connected.
26. (b) $\{z\colon |z| < 1\}$.
(c) $\{z\colon |z| = 1\} \cup [-2, -1) \cup \{2\}$.
(d) $\{2\}$ 27. E has three connected components $A = \{z\colon |z| < 1\}$, $B = [-2, -1)$ and $C = \{2\}$. Three ways to write E as a union of two separated sets is thus $E = (A \cup B) \cup C = (A \cup C) \cup B = (C \cup B) \cup A$.
30. 4.

1.2. Chapter 2

2. (a) 0.
(b) $1 + i$.
9. $T'(z) = \frac{ad-bc}{(cz+d)^2}$, it is nowhere zero.
12. (a) differentiable and holomorphic in $\mathbb{C}$ with derivative $-e^{-x}e^{-iy}$.
(b) nowhere differentiable or holomorphic.
(c) differentiable only on $\{x + iy \in \mathbb{C}\colon x = y\}$ with derivative $2x$, nowhere holomorphic.
(d) nowhere differentiable or holomorphic.
(e) differentiable and holomorphic in $\mathbb{C}$ with derivative
$-\sin(x)\cosh(y) - i\cos(x)\sinh(y)$.
(f) nowhere differentiable or holomorphic.
(g) differentiable only at 0 with derivative 0, nowhere holomorphic.
(h) differentiable only at 0 with derivative 0, nowhere holomorphic.
(i) differentiable only at i with derivative i, nowhere holomorphic.
(j) differentiable and holomorphic in $\mathbb{C}$ with derivative $2y - 2xi = -2iz$.
(k) differentiable only at 0 with derivative 0, nowhere holomorphic.
(l) differentiable only at 0 with derivative 0, nowhere holomorphic.
16. (a) $v = 2xy$.
(b) $v = \cos(x)\sinh(y)$.
(c) $v = 4xy + y$.
(d) $v = -\frac{y}{x^2+y^2}$ 18. $\frac{x}{x^2+y^2}$ is harmonic on $\mathbb{C} \setminus \{0\}$ but not on $\mathbb{C}$, $\frac{x^2}{x^2+y^2}$ is not harmonic.

1.3. Chapter 3

9. (a) $\frac{-z+1}{z-3}$ (b) $\frac{(1+i)z-(1+i)}{z-2}$.
(c) $\frac{z+i}{iz+1}$.
10. One solution is $\frac{iz-i}{z-1-i}$ sending $1 + i$ to -i.
11. $\frac{2z+1}{z+2}$.
12 (a) the lower halfplane $\{w\colon \mathrm{im}(w) < 0\}$.
(b) $\{w\colon \mathrm{Im} < 0, |w| < 1\}$.
(c) $\{w\colon \mathrm{Re}(w) < 1, |w - \frac{1}{2}| > \frac{1}{2}\}$.
14. $\frac{-1\pm\sqrt{3}}{2}$.

15. (a) $\frac{1}{1-z}$.
(b) $\frac{(1+2i)z-1}{z-1+2i}$.
(c) $\frac{1+i}{\sqrt{2}}z$.
18. $0, \infty, 1, i$, the point $(1,1,0)$ is not in the domain of the function ϕ.
20. $\frac{1}{\sqrt{3}}(1,1,-1)$, $\{z\colon |z-1-i| = \sqrt{3}\}$.
26. (a) $\{|w| = 1\}$.
(b) $\{|w| = e\}$.
(c) $\{1 \leq |w| \leq e\}$.
31. (a) -1.
(b) e^{π}.
(c) $e^{-\frac{\pi}{2}}$.
(d) $\cos(\frac{e^{-1}-e}{2}) - i\sin(\frac{e^{-1}-e}{2})$.
(e) $3+4i$.
(f) $\sqrt[4]{2}\cos(\frac{\pi}{8}) + i\sqrt[4]{2}\sin(\frac{\pi}{8})$.
(g) $\sqrt{3} - \sqrt{3}i$.
(h) -1.
32. (a) $i\frac{\pi}{2}$.
(b) $e^{-\pi}$.
(c) $\sqrt{2} + i\frac{\pi}{4}$.
33. $\mathbb{C} \setminus \big((-\infty,-1] \cup [1,\infty)\big)$.
36. (a) differentiable at 0, nowhere holomorphic.
(b) differentiable and holomorphic on $\mathbb{C} \setminus \{-1, e^{i\frac{\pi}{3}}, e^{-i\frac{\pi}{3}}\}$.
(c) differentiable and holomorphic on $\mathbb{C} \setminus \{x+iy \in \mathbb{C}\colon x \geq -1, y = 2\}$.
(d) nowhere differentiable or holomorphic.
(e) differentiable and holomorphic on $\mathbb{C} \setminus \{x+iy \in \mathbb{C}\colon x \leq 3, y = 0\}$.
(f) differentiable and holomorphic in $\mathbb{C}$ (i.e. entire).
37. (a) $z = i$.
(b) There is no solution.
(c) $z = \ln\pi + i(\frac{\pi}{2} + 2\pi k), k \in \mathbb{Z}$.
(d) $z = \frac{\pi}{2} + 2\pi k \pm 4i, k \in \mathbb{Z}$.
(e) $z = \frac{\pi}{2} + \pi k, k \in \mathbb{Z}$.
(f) $z = \pi k i, k \in \mathbb{Z}$.
(g) $z = \pi k, k \in \mathbb{Z}$.
(h) $z = 2i$.
38. $\{w\colon \mathrm{Re}(w) \in [0,1], \mathrm{Im}(w) \in (-\pi,\pi]\}$.
40. $f'(z) = cz^{c-1}$.
42. The exponentially growing spiral $\{w(t) = e^t(\cos(t) + i\sin(t)), t \in \mathbb{R}\}$, tending to infinity as t tends to positive infintity and to 0 as t tends to negative infinity.

1.4. Chapter 4

1. (a) 6.
(b) π.
(c) 4.
(d) $\frac{5^{\frac{3}{2}}-1}{3}$.
2. (a) 1.

(b) $1 + \frac{i}{3}$.
(c) $1 + i$.
3. $-2\pi i$.
4. (a) $8\pi i$.
(b) 0.
(c) 0.
(d) 0.
5. (a) $\frac{1+i}{2}$, $\frac{1+i}{2}$, i, 1.
(b) πi, $-\pi$, 0, $2\pi i$.
(c) $\pi r^2 i$, $-\pi r^2$, 0, $2\pi r^2 i$ 6. (a) $e^2(e + \cos(1) + i\sin(1))$.
(b) 0.
(c) $\frac{e^{3+3i}-1}{3}$.
7. $\frac{8}{15} + \frac{5}{6}i$.
8. -1.
9. $e^{z_0} - 1$.
10. $\frac{20+20i}{3}$.
11. $\frac{17+\ln(\frac{680}{289})}{2} + i(16 + \tan^{-1}(\frac{7}{11}))$.
12. $1 - \cosh(1)$.
13. (c) $\frac{2i}{\pi}$.
14. 2.
21. 0 if $r \in (0,1) \cup (3,\infty)$, π if $r \in (1,3)$.
23. $\frac{2\pi}{\sqrt{3}}$.
28. 0 for $r < |a|$, $2\pi i$ for $r > a$.
29. 0 for $r = 1$, $-\frac{\pi i}{3}$ for $r = 3$, 0 for $r = 5$.
31. $2\pi i$.
32. 0.
33. $-\frac{2\pi i}{3}$, $\frac{2\pi i}{3}(e^3 - 1)$

1.5. Chapter 5

1.(a) πi.
(b) $-2\pi i$.
(c) $4\pi i$.
(d) 0.
3. (a) 0.
(b) $2\pi i$.
(c) 0.
(d) πi.
(e) 0.
(f) 0.
(g) $i2\sqrt{2}\pi\sinh(\frac{1}{\sqrt{2}})$.
(h) $2\pi i e^w$ if $|w| < 3$, 0 if $|w| > 3$.
(i) $-\frac{2\pi i}{17}$.
4. $2\pi i(e^4 + 2e^2)$.
8. Any simply connected set which does not contain the origin, for example $\mathbb{C} \setminus (-\infty, 0]$.
9. (a) $(1 + i)\cosh(\frac{\pi}{2})$.

(b) $(1-i)\sinh(\pi)-(1+i)\cosh(\frac{\pi}{2})$

1.6. Chapter 6

4.(b) $-ie^z$.
5. No.

1.7. Chapter 7

2. (a) divergent.
(b) convergent (limit 0).
(c) divergent.
(d) convergent (limit $2-\frac{i}{2}$).
(e) convergent (limit 0).
3. (a) convergent.
(b) divergent.
(c) divergent.
(d) convergent.
13. 1.
20. (a) Pointwise convergence on $|z|<1$, uniform convergence on $|z|\le r$ for $r<1$.
(b) Pointwise convergence on $|z|\le 1$, uniform convergence on $|z|\le 1$.
(c) Pointwise convergence on $\mathrm{Re}(z)\ge 0$, uniform convergence on $|z|\ge r$ for $r>0$.
24. (a) $\sum_{k\ge 0}(-4)^k z^k$.
(b) $\sum_{k\ge 0}\frac{1}{3\cdot 6^k}z^k$.
(c) $\sum_{k\ge 2}\frac{k-1}{4^k}z^k$.
25. (a) $\sum_{k\ge 0}\frac{(-1)^k}{(2k)!}z^{2k}$.
(b) $\sum_{k\ge 0}\frac{(-1)^k}{(2k)!}z^{4k}$.
(c) $\sum_{k\ge 1}\frac{(-1)^{k+1}}{(2k-1)!}z^{2k+1}$.
(d) $\sum_{k\ge 1}\frac{(-1)^{k+1}2^{2k-1}}{(2k)!}z^{2k}$.
27. (a) $\sum_{k\ge 0}(-1)^k(z-1)^k$, convergence radius 1.
(b) $\sum_{k\ge 1}\frac{(-1)^{k-1}}{k}(z-1)^k$,convergence radius 1.
30. (a) ∞ if $|a|<1$, 1 if $|a|=1$, and 0 if $|a|>1$.
(b) 1.
(c) 1.
(d) 1.
(e) ∞.
(f) 1.
(g) $\frac{1}{4}$.
31. (a) $\exp(z^2)$.
(b) $\frac{1}{(2-z)^2}$.
(c) $\frac{2z^2}{(1-z)^3}$.

1.8. Chapter 8

1. (a) $\{z\in\mathbb{C}\colon |z|<1\}$, $\{z\in\mathbb{C}\colon |z|\le r\}$ for any $r<1$.
(b) $\mathbb{C}$, $\{z\in\mathbb{C}\colon |z|\le r\}$ for any r.

(c) $\{z \in \mathbb{C}\colon |z-3| > 1\}$, $\{z \in \mathbb{C}\colon r \le |z| \le R\}$ for any $1 < r \le R$.
2. (a) $\frac{2}{(1-z)^3}$.
(b) $\sinh(z)$.
(c) $1 + \frac{1}{z-4}$.
3. $\sum k \ge 0 \frac{e}{k!}(z-1)^k$.
5. $\sum_{k\ge 0} \frac{(-1)^k}{2k+1} z^{2k+1}$.
6. (a) $\frac{1}{1+z^2} = \frac{1}{2} - \frac{1}{2}(z-1) + \frac{1}{4}(z-1)^2 + 0\cdot(z-1)^3 + ...$, the convergence radius is 1.
(b) $\frac{1}{e^z+1} = \frac{1}{2} - \frac{1}{4}z + 0\cdot z^2 + \frac{1}{48}z^3 + ...$, the convergence radius is π.
10. The maximum is 3 (attained at $z = \pm i$), and the minimum is 1 (attained at $z = \pm 1$).
12. One Laurent series is $\sum_{k\ge 0}(-2)^k(z-1)^{-k-2}$, converging for $|z-1| > 2$.
13. One Laurent series is $\sum_{k\ge 0}(-2)^k(z-2)^{-k-3}$, converging for $|z-2| > 2$.
14. One Laurent series is $-3(z+1)^{-1} + 1$, converging for $z \ne -1$.
15. $\frac{1}{\sin(z)} = z^{-1} + \frac{1}{6}z + \frac{7}{360}z^3 + ...$
20. (a) $\sum_{k\ge 0} \frac{(-1)^k}{(2k)!} z^{2k-2}$.
21. $1 + \frac{z^2}{2} + \frac{5}{24}z^4 + ...$
23. (a) 1.
(b) 3.
(c) 4.
24.(a) $\pm i$, multiplicity 4.
(b) $k\pi$, $k \in \mathbb{Z}$ multiplicity 2.
(c) $(2k+1)i\pi$, $k \in \mathbb{Z}$, multiplicity 1.
(d) 0, multiplicity 3, and $\frac{\pi}{2} + k\pi$, multiplicity 1.
25. $\sum_{k\ge 0}(1 + \frac{(-1)^k}{2^{k+1}})z^k$ for $|z| < 1$, $\sum_{k\ge 0}\frac{(-1)^k}{2^{k+1}}z^k - \sum_{k<0} z^k$ for $1 < |z| < 2$ and $\sum_{k<0}(1 + (-1)^k 2^{k+1})z^k$ for $|z| > 2$.
28. It is less than or equal to $\frac{1}{2}$.
29. (a) $\sqrt{R}$.
(b) $\frac{R}{3}$.
(c) R.
(d) R.
(e) R^2.

1.9. Chapter 9

3. (a) 1, i, $-i$, of order $4, 3, 3$.
(b) $k\pi$, $k \in \mathbb{Z} \setminus \{0\}$, of order 1.
(c) 0 of order 4.
(d) $2ki\pi$, $k \in \mathbb{Z}$, of order 1.
(e) $2ki\pi$, $k \in \mathbb{Z} \setminus \{0\}$,of order 1.
7. (a) 0.
(b) 1.
(c) 4.
9. (a) One Laurent series is $\sum_{k\ge -2} \frac{(-1)^k}{4^{k+3}}(z-2)^k$, converging for $0 < |z-2| < 4$.

(b) $-\frac{\pi i}{8}$.
10. (a) $2\pi i$ (b) $\frac{27\pi i}{4}$.
(c) $-\frac{2\pi i}{17}$.
(d) $\frac{\pi i}{3}$.
(e) $2\pi i$.
(f) 0.
11. (a) $\sum_{k\geq 0} \frac{1}{ek!}(z+1^k)$.
(b) $\frac{2\pi i}{e33!}$.
13. (a) $-\frac{1}{2}$.
(b) 1.
(c) 5.
(d) e.
(e) 4.
14. (a) $\frac{1+i}{8}$.
(b) $-\frac{\pi i}{3}$.
(c) $2\pi i(1-\cos(1))$.
(d) $\frac{pii}{3}$.
15. (b) 0.
16. (a) $\frac{\pi}{2}$ for $R>1$, 0 for $R<1$.
(c) $\frac{\pi}{2}$.
17. $2\pi i \frac{f(a)-f(b)}{a-b}$.