Exam for MVE041 och MMGL32 Flervariabelmatematik

The 30 May 2015, kl. 830-1230

Help materials: Attached formula sheet. No calculators.

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Total points are 50. Passing this course requires a) 25 points of 32 points on the *Passing Part*, and b) a pass on all six Matlab labs. Your bonus points from this course apply to the passing part of the exam. The maximum score on the passing part is 32. A grade of 4 or 5 is obtained with scores of 33, and 42 respectivley. Bonus points do not apply to the mastery part of the exam.

Solutions will be posted on the course website on the first weekday following the exam. The exam is graded anonymously. Results are available on Ladok starting three weeks after the exam day. The first day on which you may contest your grade will be posted on the course website, and after that you may file a contest with the MV exp weekdays 9-13.

Passing Part

1. Consider the function $f(x, y) = \sqrt{1 + x^2/9 + y^2}$.
   (a) (1 pt) Make a representative sketch of the graph of this function.
   (b) (2 pts) Write an equation which describes the family of level curves parametrized by the constant C. Draw the $C = \sqrt{2}$ level curve, labeling the axes intercepts. Sketch other representative level curves, but do not bother to figure out the intercepts for different C-values.
   (c) (3 pts) What is the gradient of $f(x, y)$? How does the gradient relate to the level curves? Include some representative gradient vectors in the appropriate sketch above.
   (d) (1 pt) What is the equation for the tangent plane at point $P = (3, 2)$?

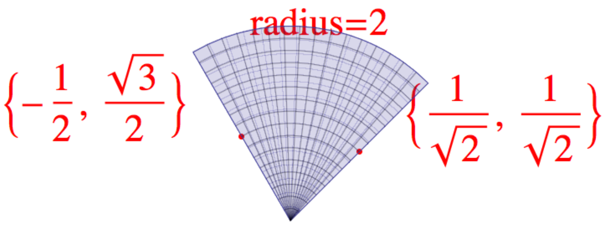
2. (4 pts) Find the parametric curve $\bar{r}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}} + z(t)\hat{\mathbf{k}}$ of intersection between the surface $z = \cos^2(x)$ and the plane $x + y + z = 1$, in terms of the parameter $t = x$. Compute $d\bar{r}(t)/dt$.

3. (4 pts) Use the Lagrange multiplier method to find the minimum distance from the origin to the curve $x^2y = 16$ in the first quadrant.

4. (4 pts) Compute the Jacobian determinant for the coordinate transformation for polar coordinates $x = r\cos\theta, y = r\sin\theta$. Evaluate

$$\iint_R 3\frac{\exp(\sqrt{x^2+y^2}/2)}{\sqrt{x^2+y^2}}dA$$

   over the region indicated in the figure.



5. (3 pts) Write the second order ODE $\frac{d^2f}{dt^2} = -1$ as a system of first order ODE. Make a representational sketch of the corresponding “phase vector field", and integral curves.

   OBS! Note that you are not asked to find explicit equations of the integral curves, only sketch them based on the vector field.

6. (4 pts) Consider the three vector fields on $\mathbb{R}^3$

$$\bar{F}(x, y, z) = -y\hat{\mathbf{i}} + x\hat{\mathbf{j}} + z\hat{\mathbf{k}}, \quad \bar{G}(x, y, z) = -y\hat{\mathbf{i}} + \hat{\mathbf{k}}, \quad \bar{H}(x, y, z) = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + 2z\hat{\mathbf{k}}.$$

   What are the divergence and curl of each vector field? Are any of the vector fields $\bar{F}, \bar{G}, \bar{H}$ solenoidal, irrotational, or conservative? Specify the vector fields that have these properties.

7. Green's theorem:

   (a) (1 pt) What is the equation of Green's theorem?

   (b) (5 pts) The parametric curve $\bar{r}(t) = \sin(t)\hat{\mathbf{i}} + \sin(2t)\hat{\mathbf{j}}$ traces out a closed path in a clock-wise direction as t goes from 0 to π. Using the vector field $\bar{F} = -y\hat{\mathbf{i}}$ and Green's theorem find the area enclosed by this curve.

Mastery Part

1. (2 pts) Evaluate $\iint_W z dS$, where W is the surface of the sphere of radius a lying below the intersection of the sphere with the cone $z = \sqrt{x^2 + y^2}$ and above the xy-plane.

2. (5 pts) Find the volume which is inside the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ and inside the cone $z^2 = \frac{c^2}{3a^2b^2}(b^2x^2 + a^2y^2)$.

3. Conservative vector fields.

   (a) (3 pts) Prove that if $\bar{F}$ is a conservative smooth vector field, and $\mathcal{C}$ is a piecewise smooth curve, then $\oint_{\mathcal{C}} \bar{F} \cdot d\bar{r} = 0$ around any closed curve $\mathcal{C}$. Also prove that if $\oint_{\mathcal{C}} \bar{F} \cdot d\bar{r} = 0$ around any closed curve $\mathcal{C}$, then $\int_{\mathcal{C}} \bar{F} \cdot d\bar{r}$ for a curve $\mathcal{C}$ from point P to point Q depends only on the end points P, Q.

   (b) (3 pts) Evaluate the line integral $\int_{\mathcal{C}} \bar{F} \cdot d\bar{r}$ for any curve from the point $P = (0, 1, 3)$ to $Q = (\pi/2, 2, 1)$, for $\bar{F} = (-\sin(x)e^y + yz)\hat{\mathbf{i}} + (\cos(x)e^y + xz)\hat{\mathbf{j}} + (z^2 + xy)\hat{\mathbf{k}}$.

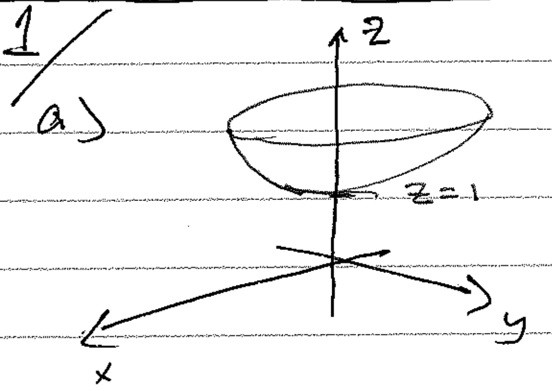
4. Divergence theorem, and electric fields with symmetry.

   (a) (2 pts) State the divergence theorem.

   (b) (3 pts) Consider a sphere of uniform charge density λ, and radius R. Maxwell's time-independent equation says that $\bar{\nabla} \cdot \bar{E} = \kappa\lambda$ for a unit-dependent constant κ, where $\bar{E}$ is the electric field. Using symmetry and the divergence theorem, find the magnitude of the electric field both inside and outside the charged sphere. Make a sketch of your result.

#1

a)



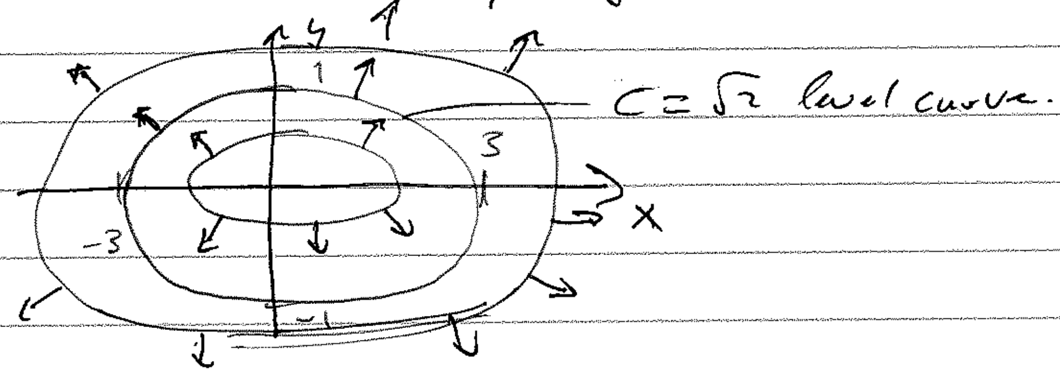
Upper portion of hyperboloid.

b) Level curves $z = f(x,y) = c$ ~ constant

$$\Rightarrow \quad c = \sqrt{1 + x^2/9 + y^2}$$

$$\Rightarrow \quad \frac{x^2}{9} + y^2 = c^2 - 1 \qquad \text{ellipses}$$

$c = \sqrt{2}$ is the $\frac{x^2}{9} + y^2 = 1$ ellipse.



c) $$\bar{\nabla} f = \frac{\frac{1}{9} x}{\sqrt{1 + x^2/9 + y^2}} \, \hat{\imath} + \frac{y}{\sqrt{1 + x^2/9 + y^2}} \, \hat{\jmath}$$

$\bar{\nabla} f$ is $\perp$ to level curves, and points in direction in which f is increasing.

d) Note $\sqrt{1 + 1 + 4} = \sqrt{6}$

$$z = \sqrt{6} + \frac{1}{3\sqrt{6}}(x - 3) + \frac{2}{\sqrt{6}}(y - 2)$$

#2

$$x + y + z = 1, \quad z = \cos^2(x)$$

$$\Rightarrow \quad x + y + \cos^2(x) = 1 \quad \Rightarrow \quad y = 1 - x - \cos^2(x)$$
$$= \sin^2(x) - x$$

$$\therefore \quad \bar{r}(x) = x\,\hat{\imath} + (\sin^2(x) - x)\,\hat{\jmath} + \cos^2(x)\,\hat{k}$$

$$\frac{d}{dx}\bar{r}(x) = 1 \cdot \hat{\imath} + (2\sin(x)\cos(x) - 1)\,\hat{\jmath}$$
$$+ 2\cos(x)\sin(x) \cdot (-1)\,\hat{k}$$

$$\boxed{\begin{aligned} \bar{r}(x) &= x\,\hat{\imath} + (\sin^2(x) - x)\,\hat{\jmath} + \cos^2(x)\,\hat{k} \\ \bar{v}(x) &= \tfrac{d}{dx}\bar{r}(x) = 1\,\hat{\imath} + (2\sin(x)\cos(x) - 1)\,\hat{\jmath} \\ &\qquad - 2\cos(x)\sin(x)\,\hat{k} \end{aligned}}$$

#3

Minimize distance squared.

Let $f(x,y) = x^2 + y^2$, $g(x,y) = x^2 y - 16$

and $L(x,y,\lambda) = f(x,y) + \lambda g(x,y)$

$$0 = \frac{\partial L}{\partial x} = 2x + 2xy\lambda \quad \Rightarrow \quad y = -1/\lambda \qquad (A)$$

$$0 = \frac{\partial L}{\partial y} = 2y + \lambda x^2 \quad \Rightarrow \quad -2y = \lambda x^2$$

So from (A) $2y^2 = x^2$ (B)

$$0 = \frac{\partial L}{\partial \lambda} = x^2 y - 16$$

$\Rightarrow$ using (B) $2y^3 = 16$, $y = 2$

$\Rightarrow$ $x^2 = 8$, $x = 2\sqrt{2}$

distance $= \sqrt{x^2 + y^2} = \sqrt{12} = 2\sqrt{3}$

$$\boxed{\text{Answer} \quad x = 2\sqrt{2}, \; y = 2, \; \text{dist.} = 2\sqrt{3}}$$

#4

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

$$\iint_R 3 \frac{e^{\frac{1}{2}\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}}\, dA = 3\iint_R \frac{e^{r/2}}{r}\, r\, dr\, d\theta$$

polar coords

Bounds $0 \le r \le 2, \quad \pi/4 \le \theta \le 2\pi/3$

$$\Rightarrow 3\int_{\theta=\frac{\pi}{4}}^{\frac{2\pi}{3}} \int_{r=0}^{2} e^{r/2}\, dr\, d\theta$$

$$= 3\left(\frac{2\pi}{3} - \frac{\pi}{4}\right) \int_0^2 2 \cdot \frac{d}{dr} e^{r/2}\, dr$$

$$= 6\left(\frac{8\pi - 3\pi}{12}\right)\left(e^1 - 1\right)$$

$$\boxed{= \frac{5\pi}{2}\left(e^1 - 1\right)}$$

#5

Let $u^1 = f$, $u^2 = \frac{d}{dt} f$

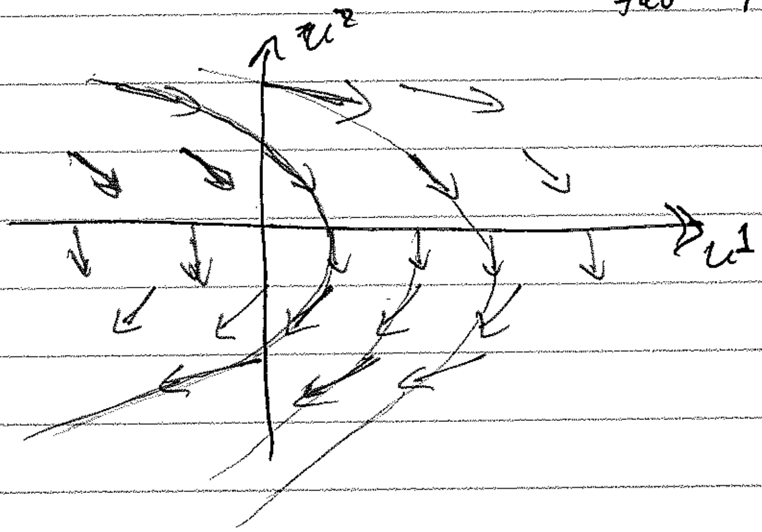
Then

$$\frac{du^1}{dt} = u^2$$

$$\frac{du^2}{dt} = -1$$

$$\Longleftrightarrow \frac{d}{dt}\bar{u}(t) = \bar{F}$$

for $\bar{F} = (u^2, -1)$



#6 $\bar{\nabla} \cdot \bar{F} = 1$, $\bar{\nabla} \bar{G} = 0$, $\bar{\nabla} \bar{H} = 4$

$$\bar{\nabla} \times \bar{F} = \hat{i} \cdot 0 - \hat{j} \cdot 0 + \hat{k}\, 2 = 2\hat{k}$$

$$\bar{\nabla} \times \bar{G} = \hat{k}$$

$$\bar{\nabla} \times \bar{H} = 0$$

$\therefore$ $\bar{G}$ is solenoidal

$\bar{H}$ is irrotational, and thus conservative.

#7

a) $\oint_C \bar{F}\cdot d\bar{r} = \iint_R \left(\frac{\partial F^2}{\partial x} - \frac{\partial F^1}{\partial y}\right) dA$

where C is the closed curve bounding R.

b) Note that for $\bar{F} = -y\hat{i}$

$$\frac{\partial F^2}{\partial x} - \frac{\partial F^1}{\partial y} = 1$$

$$\therefore \oint_C \bar{F}\cdot d\bar{r} = -\iint_R 1\, dA = -\text{Area}(R)$$

desired area

"$-$" for clockwise oriented curve!

Compute line integral

$$\bar{F}\cdot d\bar{r} = F^1 dx + F^2 dy, \quad x(t) = \sin(t),\ y(t) = \sin(2t)$$

$$= -y\,dx + 0\cdot dy$$

$$= -\sin(2t)\cos(t)\,dt$$

$$= -2\sin(t)\cos^2(t)\,dt$$

$$= -2\sin(t)\left(1-\sin^2(t)\right)dt$$

$$\therefore \text{Area}(R) = 2\int_0^{\pi} \sin(t)\,dt - 2\int_0^{\pi} \sin^3(t)\,dt$$

$$= 4 - 2\left(-\frac{1}{3}\sin^2(t)\cos(t)\Big|_0^{\pi} - \frac{2}{3}\cos(t)\Big|_0^{\pi}\right)$$

$$= 4 - \frac{4}{3}\cdot 2 = \underline{\underline{4/3}}$$

$$\boxed{\text{Area}(R) = 4/3}$$

Mastery Section

M1.

$$\int_{\varphi=\pi/4}^{\pi/2}\int_{\theta=0}^{2\pi} z\Big|_{\text{Sphere}} a^2 \sin\varphi \, d\varphi \, d\theta = 2\pi \int_{\varphi=\pi/4}^{\pi/2} a^3 \cos\varphi \sin\varphi \, d\varphi$$

$$= 2\pi a^3 \int_{\pi/4}^{\pi/2} \frac{1}{2}\frac{d}{d\varphi} \sin^2\varphi \, d\varphi$$

$$= \pi a^3 \left(1 - 1/2\right)$$

$$\boxed{= \pi a^3/2}$$

M.2 Let $x = au$, $y = bv$, $z = cw$

Then

$u^2 + v^2 + w^2 = 1$ and $c^2 w^2 = \frac{c^2}{3a^2b^2}\left(a^2b^2u^2 + a^2b^2v^2\right)$

sphere rad. 1 $\qquad \Rightarrow w^2 = \frac{1}{3}\left(u^2 + v^2\right)$

$$\text{Vol} = \iiint_R dx\,dy\,dz = \iiint_D J(u,v,w)\, du\, dv\, dw \quad , J(u,v,w) = abc$$

Introduce spherical coords in u, v, w - space. Call them (ρ, φ, θ)

$$\Rightarrow \text{Vol} = abc2 \int_{\varphi=0}^{\varphi_c}\int_{\rho=0}^{1}\int_{\theta=0}^{2\pi} \rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta \quad , \varphi_c = \text{ArcTan}(\sqrt{3}) = \pi/3$$

$$= 4\pi abc \, \frac{1}{3}\left(-\cos\varphi\right)\Big|_0^{\pi/3}$$

$$= \frac{4}{3}\pi abc\left(-1/2 + 1\right)$$

$$\boxed{= \frac{2}{3}\pi abc}$$

M3

a) $\bar{F}$ is conservative $\Rightarrow$ $\bar{F} = \bar{\nabla}\phi$

C is piecewise smooth with parameterization $\bar{r}(t)$, $a \le t \le b$

then C is closed $\Rightarrow$ $\bar{r}(a) = \bar{r}(b)$.

$$\oint_C \bar{F}\cdot d\bar{r} = \oint_C \bar{\nabla}\phi(\bar{r})\cdot d\bar{r}$$

$$= \oint_C \frac{\partial\phi(\bar{r})}{\partial x}dx + \frac{\partial\phi(\bar{r})}{\partial y}dy + \frac{\partial\phi(\bar{r})}{\partial z}dz$$

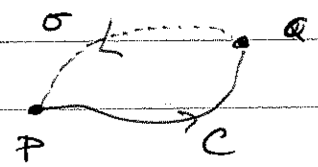
$$= \int_a^b \left\{ \frac{\partial\phi(\bar{r}(t))}{\partial x}\frac{dx}{dt}dt + \frac{\partial\phi(\bar{r}(t))}{\partial y}\frac{dy}{dt}dt + \frac{\partial\phi(\bar{r}(t))}{\partial z}\frac{dz}{dt}dt \right\}$$

$$= \int_a^b \frac{d}{dt}\phi(\bar{r}(t))\,dt \text{, by chain rule.}$$

$$= \phi(\bar{r}(b)) - \phi(\bar{r}(a))$$

$$= 0 \qquad \square$$

Second Proof. Let C be any curve from P to Q



Let σ be any curve from Q to P

Then $C + \sigma$ is a closed curve

$$0 = \oint_{C+\sigma} \bar{F}\cdot d\bar{r} = \int_C \bar{F}\cdot d\bar{r} + \int_\sigma \bar{F}\cdot d\bar{r}$$

Thus for any two curves C and σ, depending only on the end points we have

$$\int_{-\sigma} \bar{F}\cdot d\bar{r} = -\int_\sigma \bar{F}\cdot d\bar{r} = \int_C \bar{F}\cdot d\bar{r} \qquad \square$$

M3/

b) Find a potential $\phi(x,y,z)$.

$$\phi_{test}(x,y,z) = \int F^1\, dx = \int \left(-\sin(x) e^y + yz\right) dx$$

$$= \cos(x)\, e^y + xyz + h(y,z)$$

Now compare $F^2 = \cos(x) e^y + xz$

with

$$\frac{\partial \phi_{test}}{\partial y} = \cos(x) e^y + xz + \frac{\partial}{\partial y} h(y,z)$$

$$\Rightarrow \frac{\partial}{\partial y} h(y,z) = \mathbb{C} \quad (\text{constant})$$

$$\therefore \phi_{test}(x,y,z) = \cos(x) e^y + xyz + g(z)$$

Now Compare $F^3 = z^2 + xy$

with

$$\frac{\partial \phi_{test}}{\partial z} = xy + \frac{\partial}{\partial z} g(z)$$

$$\Rightarrow \frac{\partial}{\partial z} g(z) = z^2 \Rightarrow g(z) = \frac{1}{3} z^3$$

$$\therefore \phi(x,y,z) = \cos(x) e^y + xyz + \frac{1}{3} z^3$$

$$\int_C \bar{F} \cdot d\bar{r} = \phi\left(\frac{\pi}{2}, 2, 1\right) - \phi(0,1,3)$$

$$= \pi + \frac{1}{3} - (e + 9)$$

$$= \pi - e - \frac{26}{3}$$

M4

a) Let D be a regular domain in $\mathbb{R}^3$ with boundary S which is an oriented, closed surface with normal field $\hat{N}$ in the outward direction. If $\bar{F}$ is a smooth vector field on D, then

$$\iiint_D \operatorname{div} \bar{F}\, dV = \oiint_S \bar{F} \cdot \hat{N}\, dS$$

b) By symmetry $\bar{E}$ points in the radial direction.

Inside: Let B be the ball of radius $\rho < R$ with boundary S. The outward normal to S is the radial unit vector $\hat{N}$.

$$\text{Then,} \quad \iiint_B \operatorname{div} \bar{E}\, dV = \iiint_B k\lambda\, dV = k\lambda \frac{4}{3}\pi\rho^3$$

$$\text{and} \quad \oiint_S \bar{E} \cdot \hat{N}\, dS = |\bar{E}| \oiint_S dS = |\bar{E}|\, 4\pi\rho^2$$

$$\therefore \quad |\bar{E}|_{inside} = \frac{k\lambda}{3}\rho$$

Outside: Same set up, but now $\rho > R$.

$$\iiint_B \operatorname{div} \bar{E}\, dV = k\lambda \frac{4}{3}\pi R^3$$

$$\oiint_S \bar{E} \cdot \hat{N}\, dS = |\bar{E}|\, 4\pi\rho^2$$

$$\therefore \quad |\bar{E}|_{outside} = \frac{k\lambda}{3} \frac{R^3}{\rho^2}$$