Tentamen
MVE041 Flervariabelanalys

2015-05-25 kl. 8.30–12.30

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Telefonvakt: To be announced , telefon: ingen medskip
Hjälpmedel: bifogat formelblad, ordlistan från kurswebbsidan, ej räknedosa

For the passing section 25 points of 32 are needed. Your bonus points from this course apply to the passing part. The maximum score on the passing part is 32.
For the mastery section, a grade of 4 is achieved with a score of 33 - 41 points, and a grade 5 with a score of 42 - 50 points. Bonus points do not apply the mastery part of the exam.

Formelblad för MVE041, 14/15

Trigonometri.

$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$$

$$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$$

$$\cos(x)\cos(y) = \frac{1}{2}(\cos(x-y) + \cos(x+y))$$

$$\sin(x)\sin(y) = \frac{1}{2}(\cos(x-y) - \cos(x+y))$$

$$\sin(x)\cos(y) = \frac{1}{2}(\sin(x-y) + \sin(x+y))$$

$$\tan(x+y) = \frac{\tan(x)+\tan(y)}{1-\tan(x)\tan(y)}$$

Integralkatalog

$$\int x^a\,dx = \frac{x^{a+1}}{a+1} + C \quad , \quad a \neq -1$$

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int \sin x\,dx = -\cos x + C$$

$$\int \cos x\,dx = \sin x + C$$

$$\int \frac{1}{cos^2 x}\,dx = \tan x + C$$

$$\int \frac{1}{sin^2 x}\,dx = -\cot x + C$$

$$\int e^x\,dx = e^x + C$$

$$\int a^x\,dx = \frac{a^x}{\ln a} + C \ , \ 0 < a \neq 1$$

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\arctan\frac{x}{a} + C \ , \ a \neq 0$$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$$

$$\int \frac{1}{\sqrt{a-x^2}}\,dx = \arcsin\frac{x}{\sqrt{a}} + C \ , \ a > 0$$

$$\int \sqrt{a-x^2}\,dx = \frac{1}{2}x\sqrt{a-x^2} + \frac{a}{2}\arcsin\frac{x}{\sqrt{a}} + C \ , \ a > 0$$

$$\int \frac{1}{\sqrt{x^2+a}}\,dx = \ln|x+\sqrt{x^2+a}| + C \ , \ a \neq 0$$

$$\int \sqrt{x^2+a}\,dx = \frac{1}{2}(x\sqrt{x^2+a} + a\ln|x+\sqrt{x^2+a}|) + C$$

$$\int \frac{1}{(x^2+1)^n}dx = I_n$$

$$I_{n+1} = \frac{x}{2n(x^2+1)^n} + \frac{2n-1}{2n}I_n$$

Maclaurinutvecklingar

$$e^x = \sum_{k=0}^{\infty}\frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

$$\sin x = \sum_{k=1}^{\infty}(-1)^{k-1}\frac{x^{2k-1}}{(2k-1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

$$\cos x = \sum_{k=0}^{\infty}(-1)^k\frac{x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

$$(1+x)^\alpha = \sum_{k=0}^{\infty}\binom{\alpha}{k}x^k = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \ldots \quad , \quad |x| < 1 \quad , \quad \binom{\alpha}{k} = \frac{\alpha(\alpha-1)\ldots(\alpha-k+1)}{k(k-1)\ldots 1}$$

$$\ln(1+x) = \sum_{k=1}^{\infty}(-1)^{k+1}\frac{x^k}{k} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \quad , \quad -1 < x \leq 1$$

$$\arctan x = \sum_{k=1}^{\infty}(-1)^{k-1}\frac{x^{2k-1}}{2k-1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots \quad , \quad |x| \leq 1$$

Passing Part

The questions in this section are each worth 4 points, unless otherwise indicated.

1. Consider the function $f(x, y) = 1 - x^2 - y^2/4$.

(a) (1 pt) Make a representative sketch of the graph of this function, some level curves, and the gradient vector field. In your sketch label important features.

(b) (2 pts) At the point $P = (1/2, 1)$ what is the gradient of $f(x, y)$? What is the magnitude of the gradient at P?

(c) (1 pt) What is the equation for the tangent plane at point P?

2. What is the rectangle with the largest area that can be inscribed inside the ellipse

$$x^2/4 + y^2/9 = 1?$$

By *inscribed* we mean that the corners of the rectangle are constrained to lie on the ellipse.

3. (a) Compute the Jacobian determinant for cylindrical coordinates

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z.$$

(b) Compute the volume of the region bounded by $x^2 + y^2 \leq 1$ and $-\sqrt{x^2 + y^2} \leq z \leq 2 - x^2 - y^2$.

4. Write the second order ODE

$$\frac{d^2 f}{dt^2} = -kf$$

as a system of first order ODE. Make a representational sketch of the corresponding "phase vector field", and integral curves. Describe the behavior of the system. Note that you are not asked to find explicit equations of the integral curves, only sketch them based on the vector field.

5. Integrate the vector field $\bar{F}(x, y, z) = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ along the circular helix described by $\bar{r}(t) = 6\cos(t)\hat{\mathbf{i}} + 6\sin(t)\hat{\mathbf{j}} + 2t\hat{\mathbf{k}}$ from $t = 0$ to $t = 2\pi$.

6. (a) What is the definition of a conservative vector field $\bar{F}$?

(b) **True** or **False**: If $\bar{F}$ is a conservative vector field, and $\mathcal{C}$ is a closed path, then $\oint_{\mathcal{C}} \bar{F} \cdot d\bar{r} = 0$.

(c) Let $\bar{F}$ be a conservative vector field. What is $curl \bar{F}$?

(d) The gradient of a function is orthogonal to the level curves of the function. What does this imply about the field lines for a conservative vector field?

7. Evaluate $\oint_C \bar{F} \cdot d\bar{r}$ for

$$\bar{F}(x, y) = \left(-3 + y\sqrt{x^2 + y^2}\right)\hat{\mathbf{i}} + \left(3 - x\sqrt{x^2 + y^2}\right)\hat{\mathbf{j}},$$

and where C is the closed curve with clockwise orientation bounding the region defined by $y > x, y > -x$ and $y < \sqrt{a^2 - x^2}$

8. (a) (1 pts) What is the equation of the divergence theorem?

(b) (3 pts) Verify the divergence theorem for the vector field $\bar{F} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ over the ball $x^2 + y^2 + z^2 \leq 9$.

Mastery Part

9. (a) (3 pts) Suppose the function $f(x, y)$ is differentiable at the point (a, b), and that $\nabla f(a, b) \neq 0$. Prove that $\nabla f(a, b)$ is normal to the level curve of f which passes through (a, b).

(b) (3 pts) Give the definition of the directional derivative, $D_{\bar{u}} f(x, y)$. Prove that the directional derivative of $f(x, y)$ in the direction $\bar{u} = (u_1, u_2)$ is given by $D_{\bar{u}} f(x, y) = \bar{u} \cdot \bar{\nabla} f(x, y)$.

10. (6 pts) Integrate the function $f(x, y) = 12x^3 y + 24xy^3$ over the domain bounded by the curves $y^2 \geq 1 + x^2, y^2 \leq 3 + x^2$, and $1 \leq 2x^2 + y^2 \leq 4$.

11. (6 pts) Find the area of the oblate spheroid $x^2/16 + y^2/16 + z^2/1 = 1$ which is contained inside the cylinder $x^2 + y^2 = 4x$. (*Hint: to avoid really nasty integration try a change of variables.*)