Exam for MVE041 och MMGL32 Flervariabelmatematik

May 25, 2015

Hjälpmedel: This cover sheet, inga räknare.

Totalpoäng 50. A passing grade is achieved by scoring 25 of 32 points on the passing part.

Grade 4 by 33-41, grade 5 with 42-50.

1 Passing Part

The questions in this section are each worth 4 points, unless otherwise indicated.

1. Consider the function $f(x, y) = 1 - x^2 - y^2/4$.

   (a) (1 pt) Make a representative sketch of the graph of this function, some level curves, and the gradient vector field. In your sketch label important features.

   (b) (2 pts) At the point $P = (1/2, 1)$ what is the gradient of $f(x, y)$? What is the magnitude of the gradient at P?

   (c) (1 pt) What is the equation for the tangent plane at point P?

   Solution:

   (a) The graph is an downward facing elliptic paraboloid. At $z = 0$ the x, y intercepts are $1, 2$ respectively. The level curves satisfy the equation $f(x, y) = C \implies x^2 + y^2/4 = 1 - C$. Thus, these are ellipses, with C defined for 1 through $-\infty$. You could sketch the $C = 0$ level curve, or the $C = -3$ level curve, which in this case are nice ellipses in the xy-plane.

   The important point about the sketching the gradient vector field is that a) it lies in the xy-plane, and b) it is orthogonal to the level curves, and c) that it points "up hill".

   (b) $\bar{\nabla} f = -2x\hat{\mathbf{i}} - 1/2y\hat{\mathbf{j}}$. So at P $\bar{\nabla} f(1/2, 1) = -1\hat{\mathbf{i}} - 1/2\hat{\mathbf{j}}$, and $|\bar{\nabla} f(1/2, 1)| = \sqrt{5}/2$.

   (c) $z = -1(x - 1/2) - 1/2(y - 1) + 1/2$.

2. What is the rectangle with the largest area that can be inscribed inside the ellipse

$$x^2/4 + y^2/9 = 1?$$

   By *inscribed* we mean that the corners of the rectangle are constrained to lie on the ellipse.

   Solution:
   Maximize $A(x, y) = xy$ subject to the constraint $x^2/a^2 + y^2/b^2 = 1$. Argue that area will be four times area in first quadrant by symmetry. Use Lagrange multipliers to get $x = a/\sqrt{2}, y = b/\sqrt{2}$ and thus $A_{\text{first quadrant}} = ab/2 \implies A = 2ab$.

3. (a) Compute the Jacobian determinant for cylindrical coordinates

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z.$$

(b) Compute the volume of the region bounded by $x^2 + y^2 \leq 1$ and $-\sqrt{x^2+y^2} \leq z \leq 2 - x^2 - y^2$.

> *Solution:*
>
> (a) You should find as we did in class that $J(r,\theta,z) \equiv \frac{\partial(x,y,z)}{\partial(r,\theta,z)} = r$.
>
> (b) Several ways to find this volume. For instance you can integrate in z from $-r$ to $2 - r^2$, and then in r from 0 to 1 and in θ from 0 to 2π. Result: $13\pi/6$.

4. Write the second order ODE

$$\frac{d^2 f}{dt^2} = -kf$$

as a system of first order ODE. Make a representational sketch of the corresponding "phase vector field", and integral curves. Describe the behavior of the system. Note that you are not asked to find explicit equations of the integral curves, only sketch them based on the vector field.

> *Solution:* This is the problem illustrated in the notes on second order ODE and phase space on the website. Introduce new variables u^1 and u^2 defined by
>
> $$u^1(t) = f(t), \quad u^2(t) = \frac{df(t)}{dt}. \tag{1}$$
>
> From the equation for f we obtain
>
> $$\frac{du^1(t)}{dt} = u^2(t), \tag{2}$$
>
> $$\frac{du^2(t)}{dt} = -ku^1(t), \tag{3}$$
>
> or in vector form
>
> $$\frac{d}{dt}\bar{\mathbf{u}}(t) = \bar{\mathbf{F}}(\bar{\mathbf{u}}), \tag{4}$$
>
> where $\bar{\mathbf{F}}(\bar{\mathbf{u}}) = (u^2, -ku^1)$.
> See the notes for phase vector field, integral curves and discussion!

5. Integrate the vector field $\bar{F}(x,y,z) = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ along the circular helix described by $\bar{r}(t) = 6\cos(t)\hat{\mathbf{i}} + 6\sin(t)\hat{\mathbf{j}} + 2t\hat{\mathbf{k}}$ from $t = 0$ to $t = 2\pi$.

> *Solution:* $\int_{Helix} \bar{F} \cdot d\bar{r} = \int_0^{2\pi} 4t dt = 8\pi^2$

6. (a) What is the definition of a conservative vector field $\bar{F}$?

(b) **True** or **False**: If $\bar{F}$ is a conservative vector field, and $\mathcal{C}$ is a closed path, then $\oint_{\mathcal{C}} \bar{F} \cdot d\bar{r} = 0$.

(c) Let $\bar{F}$ be a conservative vector field. What is $curl \bar{F}$?

(d) The gradient of a function is orthogonal to the level curves of the function. What does this imply about the field lines for a conservative vector field?

Solution:

(a) A vector field $\bar{F}$ is conservative if there exists a potential ϕ such that $\bar{F} = \bar{\nabla}\phi$.

(b) True

(c) zero

(d) The integral curves of a conservative vector field with potential ϕ are orthogonal to the level curves of ϕ.

7. Evaluate $\oint_C \bar{F} \cdot d\bar{r}$ for

$$\bar{F}(x,y) = \left(-3 + y\sqrt{x^2+y^2}\right)\hat{\mathbf{i}} + \left(3 - x\sqrt{x^2+y^2}\right)\hat{\mathbf{j}},$$

and where C is the closed curve with clockwise orientation bounding the region defined by $y > x, y > -x$ and $y < \sqrt{a^2 - x^2}$

Solution: Since the curve is in the clockwise direction, we have by Green's theorem (note the minus sign)

$$\oint_{\mathcal{C}} \bar{F} \cdot d\bar{r} = -\int\int_R (\partial F^2/\partial x - \partial F^1/\partial y) dA$$

Because of the symmetry of the domain and the form of the integrand we use polar coordinates. We find $\int_{\theta=\pi/4}^{3\pi/4} \int_{r=0}^{a} r^2 dr d\theta = a^3\pi/2$.

8. (a) (1 pts) What is the equation of the divergence theorem?

(b) (3 pts) Verify the divergence theorem for the vector field $\bar{F} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ over the ball $x^2 + y^2 + z^2 \leq 9$.

Solution:

(a) $\iiint_R div\bar{F} dV = \oiint_{\mathcal{S}} \bar{F} \cdot \hat{N} dS$.

(b) For the left hand side we have: $div\bar{F} = 3$, so $\iiint_R div\bar{F} dV = 3vol(ball) = 4\pi * 27$. For the right hand side we have: $\hat{N} = \frac{x\hat{\mathbf{i}}+y\hat{\mathbf{j}}+z\hat{\mathbf{k}}}{\sqrt{x^2+y^2+z^2}} = 1/3(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}})$ on the sphere of radius 3. Thus, $\bar{F} \cdot \hat{N} = 3$, and $\oiint_{\mathcal{S}} \bar{F} \cdot \hat{N} dS = 4\pi * 27$.

2 Mastery Part

9. (a) (3 pts) Suppose the function $f(x, y)$ is differentiable at the point (a, b), and that $\nabla f(a, b) \neq 0$. Prove that $\nabla f(a, b)$ is normal to the level curve of f which passes through (a, b).

   (b) (3 pts) Give the definition of the directional derivative, $D_{\bar{u}} f(x, y)$. Prove that the directional derivative of $f(x, y)$ in the direction $\bar{u} = (u_1, u_2)$ is given by $D_{\bar{u}} f(x, y) = \bar{u} \cdot \bar{\nabla} f(x, y)$.

> *Tips:*
> a) Parametrize the curve. Use the definition of a level curve, and the chain rule.
> b) From definition and chain rule.

10. (6 pts) Integrate the function $f(x, y) = 12x^3y + 24xy^3$ over the domain bounded by the curves $y^2 \geq 1 + x^2, y^2 \leq 3 + x^2$, and $1 \leq 2x^2 + y^2 \leq 4$.

> *Tips:*
> Try the change of variables $u = 2x^2 + y^2$ and $v = y^2 - x^2$. Answer: 27

11. (6 pts) Find the area of the oblate spheroid $x^2/16 + y^2/16 + z^2/1 = 1$ which is contained inside the cylinder $x^2 + y^2 = 4x$. (*Hint: to avoid really nasty integration try a change of variables.*)

> *Tips:*
> Try the change of variables $x = 4u, y = 4v, z = w$. Then the equations simplify to that of a sphere radius one, and the cylinder $u^2 + v^2 = u$. The Jacobian determinant of the transformation is 16. The surface element can be computed $dS = dudv/\sqrt{1 - u^2 - v^2}$. Introducing polar coordinates in uv-plane $u = r\cos(\theta), v = r\sin(\theta)$, we end up evaluating the integral $area = 4 * 16 * \int_{\theta=-\pi/2}^{0} \int_{r=0}^{\cos(\theta)} \frac{rdrd\theta}{\sqrt{1-r^2}}$. The factor of 4 comes from the fact that we are computing one quarter of the area with this integral. Integrating we get $64(1 + \pi/2)$.