Tentamen i Flervariabelmatematik Z MVE041 den 15 jan -14 kl 8.30-12.30

Hjälpmedel: BETA, inga räknare Telefon: Jacob Hultgren 0703-088304 Om inget annat anges är varje uppgift värd 6p. Totalpoäng 50 betygsgränser 20, 30 och 40

1) I vilken riktning är tillväxten störst för $f(x,y) = e^{xy}xy$ i punkten (1,2) och hur stor är den ? Hur stor är den i riktningen (1,1) i samma punkt? Vad är ekvationen för tangentplanet i samma punkt ? (8p)
2) Hur skall differentialekvationen $x'' + x' + x = 2$ presenteras för ode45 ? (4p)
3) Vi vill dra en kurva $y = a + be^{cx}$ genom punkterna (0, 1.0), (1, 1.2) och (2, 3.4). Skriv upp ett ekvationssystem för a,b och c och iterationsformeln för att lösa det med Newtons metod.
4) Beräkna $\iint_D x^2y\,dxdy$ där D är triangeln med hörn i (0,0), (1,1) och (0,1)
5) Beräkna volymen av området beskrivet av $x^2 + y^2 \le z \le \sqrt{x^2 + y^2}$
6) Finn en potential till fältet $(P, Q) = (e^{x^2y}2xy + 2x, e^{x^2y}x^2 + 2)$ och beräkna $\int_{(1,0)}^{(2,2)} Pdx + Qdy$
7) Bestäm avståndet från (2,3,0) till dubbelkonen $z^2 = x^2 + y^2$
8) Lös t ex genom att först göra variabelbytet $u = x + y \quad v = xy$ ekvationen $f'_x - f'_y = y - x$ Bestäm speciellt den lösning som uppfyller $f(x,0) = e^x$ (8p)

MVE041 2014-01 Exam

1a) In which direction is the growth largest for $f(x,y) = e^{xy}xy$ at the point $(1,2)$ and how large is it?

b) How large is the growth in the direction of $\bar{v} = e_1 + e_2$ at the same point?

c) What is the eq'n for the tangent plane at that point?

Solution

a) Compute ∇f at $p = (1,2)$

$$f_1(x,y) = ye^{xy}xy + ye^{xy}$$

$$\Rightarrow f_1(1,2) = 4e^2 + 2e^2 = 6e^2$$

$$f_2(x,y) = xe^{xy}xy + xe^{xy}$$

$$\Rightarrow f_2(1,2) = 2e^2 + e^2 = 3e^2$$

$$\therefore \boxed{\nabla f(1,2) = (6e^2, 3e^2)} = 3e^2(2,1)$$

$$|\nabla f(1,2)| = e^2\sqrt{45} = 3e^2\sqrt{5}$$

b) Compute $D_{\bar{u}}f$, $\bar{u} = 1/\sqrt{2}\,(1,1)$.

$$D_{\bar{u}}f = \bar{u}\cdot\nabla f = \left(6/\sqrt{2} + 3/\sqrt{2}\right)e^2 \boxed{= 9\, e^2/\sqrt{2}}$$

c) $$z = f(1,2) + f_1(1,2)(x-1) + f_2(1,2)(y-2)$$

$$= 2e^2 + 6e^2(x-1) + 3e^2(y-2)$$

$$= -10e^2 + 6e^2x + 3e^2y$$

2 How shall the differential equation

$$x''(t) + x'(t) + x = 2$$

be presented for ode45 ?

Solution

Ode45 deals with first order system.

Let $y = x'$

We then get the system

$$\begin{cases} x' = y \\ y' + y + x = 2 \end{cases} \sim \begin{cases} U_1' = U_2 \\ U_2' = 2 - U_1 - U_2 \end{cases}$$

$$[t, U] = \text{ode45}('F', [t_0, t_f], [U_{10}, U_{20}])$$

with

$$F = @(t, U)[\,U(2),\ 2 - U(1) - U(2)\,]$$

3 We want to draw a curve $y = a + be^{cx}$ through the points $(0, 1.0)$, $(1, 1.2)$, and $(2, 3.4)$. Write up a system of eq'ns for a, b, c and and iteration formula for the solution using Newton's method.

Solution

Let $\bar{p} = (a, b, c)$, We want to find a curve $y(x)$ satisfying

$$y(0) = a + b = 1.0$$

$$y(1) = a + be^{c} = 1.2$$

$$y(2) = a + be^{2c} = 3.4$$

Define functions

$$f^1(\bar{p}) = a+b-1$$
$$f^2(\bar{p}) = a+be^c - 1.2$$
$$f^3(\bar{p}) = a+be^{2c} - 3.4$$

and $\bar{f} = (f^1, f^2, f^3)$. Thus the curve satisfies $\bar{f}(\bar{p}) = 0$. Our aim is to find the root $\bar{p} = (a,b,c)$ of this expression.

The linearization about $\bar{P}_n$ of $\bar{f}(\,)$ is

$$L(\bar{P}_{n+1}) = \bar{f}(\bar{P}_n) + D\bar{f}(\bar{P}_n)(\bar{P}_{n+1} - \bar{P}_n)$$

In Newton's method we find successive values of $\bar{P}_{n+1}$ such that $L(\bar{P}_{n+1}) = 0$, or

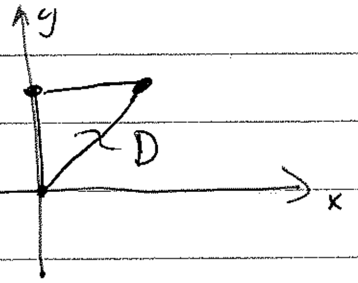
$$\bar{P}_{n+1} = \bar{P}_n - (D\bar{f}(\bar{P}_n))^{-1}\bar{f}(\bar{P}_n)$$

The main task is to compute $(D\bar{f})^{-1}$

$$D\bar{f}(\bar{p}) = \begin{pmatrix} \frac{\partial f^1}{\partial p^1} & \frac{\partial f^1}{\partial p^2} & \frac{\partial f^3}{\partial p^3} \\ \frac{\partial f^2}{\partial p^1} & \frac{\partial f^2}{\partial p^2} & \frac{\partial f^2}{\partial p_3} \\ \frac{\partial f^3}{\partial p^1} & \frac{\partial f^3}{\partial p^2} & \frac{\partial f^3}{\partial p_3} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & e^c & be^c \\ 1 & e^{2c} & 2be^{2c} \end{pmatrix}$$

#4 Compute $\iint_D x^2 y \, dx\,dy$ where D is the triangle with corners $(0,0)$, $(1,1)$, and $(0,1)$.

Solution



$$I = \iint_D x^2 y \, dx\,dy$$

$$= \int_{y=0}^{1} \int_{x=0}^{y} x^2 y \, dx\, dy$$

$$\therefore \; I = \int_{y=0}^{1} y \left(\tfrac{1}{3} x^3 \right)_0^y dy$$

$$= \frac{1}{3} \int_0^1 y^4 \, dy \qquad \boxed{= \tfrac{1}{15}}$$

#5 Compute the volume of the region described by $x^2 + y^2 \le z \le \sqrt{x^2+y^2}$

Solution

Let $r^2 = x^2 + y^2$.

The region is above the circular paraboloid and below the cone. These intersect at $r=0$, and $r=1$.

Thus in cyl coords

$$\text{Volume} = \int_{\varphi=0}^{2\pi} \int_{r=0}^{1} \int_{z=r^2}^{r} r \, dr\, dz\, d\varphi = 2\pi \int_{r=0}^{1} \int_{r^2}^{r} r \, dr\, dz$$

$$= 2\pi \int_{r=0}^{1} r(r - r^2)\, dr = 2\pi \left(\frac{1}{3} - \frac{1}{4} \right)$$

$$\boxed{= \pi/6}$$

#6 Find a potential ϕ to the field

$$(P,Q) = \left(e^{x^2y}\,2xy + 2x,\ e^{x^2y}\,x^2 + 2\right)$$

and Compute $\displaystyle\int_{(1,0)}^{(2,2)} P\,dx + Q\,dy$.

Solution

Plan: Find ϕ s.t. $\bar{\nabla}\phi(x,y) = \bar{F}(x,y) \equiv (P,Q)$.

Use $\displaystyle\int_{P_1}^{P_2} \bar{\nabla}\phi \cdot d\bar{r} = \phi(P_2) - \phi(P_1)$.

If $\bar{F}$ is conservative, then $\exists\ \phi : \bar{\nabla}\phi = \bar{F}$.

$$\Rightarrow \quad F^1 = \partial_x \phi - h(y)$$
$$F^2 = \partial_y \phi - g(x).$$

$$\begin{aligned}
\phi &= \int F^1\,dx + h(y) \\
&= \int (e^{x^2y}\,2xy + 2x)\,dx + h(y) \\
&= \int \partial_x (e^{x^2y})\,dx + x^2 + h(y) \\
&= e^{x^2y} + x^2 + h(y)
\end{aligned}$$

$$\begin{aligned}
\phi &= \int F^2\,dy + g(x) \\
&= \int (e^{x^2y}\,x^2 + 2)\,dy + g(x) \\
&= e^{x^2y} + 2y + g(x)
\end{aligned}$$

$\therefore\ \phi = e^{x^2y} + x^2 + 2y$ up to a constant.

$$\int_{(1,0)}^{(2,2)} P\,dx + Q\,dy = \int_{(1,0)}^{(2,2)} \bar{\nabla}\phi \cdot d\bar{r} = \phi(2,2) - \phi(1,0)$$

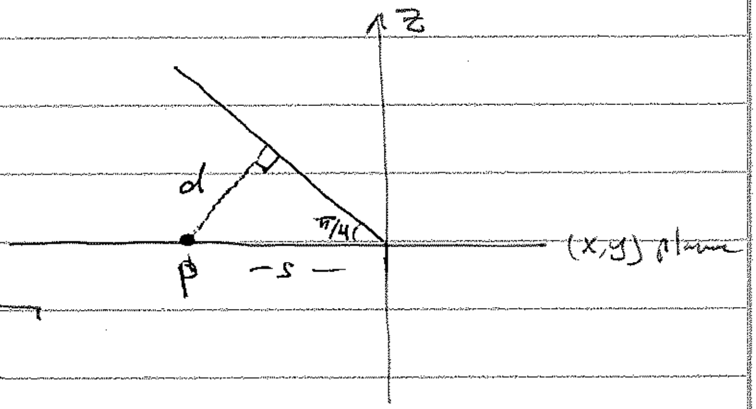
$$= e^8 + 4 + 4 - (1 + 1)$$

$$\boxed{= e^8 + 6}$$

#7 Compute the distance from $p = (2,3,0)$ to the double cone $z^2 = x^2 + y^2$.

Solution

Geometric Solution



$d = s \cdot \sin(\pi/4)$

$= \sqrt{13}\ \sqrt{2}/2$

$= \underline{\sqrt{26}/2} = \sqrt{13/2}$

Should get the same answer via Lagrange multipliers.

#8

$$f'_x - f'_y = y - x$$

$$f_1 - f_2 = \cancel{y-x}$$

$$g(u,v) = f(x,y) \quad , x =$$

$$\frac{\partial u}{\partial x} = 1 , \quad \frac{\partial u}{\partial y} = +1 \quad , \frac{\partial v}{\partial x} = y , \frac{\partial v}{\partial y} = x$$

$$\frac{\partial f}{\partial x} = \frac{\partial g}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial g}{\partial v}\frac{\partial v}{\partial x}$$

$$= \partial_u g + \partial_v g \, y$$

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial g}{\partial v}\frac{\partial v}{\partial y}$$

$$= \partial_u g + \partial_v g \, x$$

$$f_x - f_y = x - y \quad \Rightarrow \partial_u g + \partial_v g - \partial_u g - \partial_v g = x - y$$

$$\Rightarrow \partial_v g = \frac{x-y}{y-x} = -1$$

$$g(u,v) = f(x(u,v), y(u,v))$$

$$\boxed{f(x,y) = xy + g(x+y)}$$