Lösningar till tentamen MVE045, 20/10 2009

① a) $\lim_{x\to 0+} \dfrac{e^{-\frac{1}{x}}}{x^2}$:

Lösning: $\dfrac{e^{-\frac{1}{x}}}{x^2} = \left\{ \text{sätt } t=\frac{1}{x},\ x\to 0+ \Leftrightarrow t\to +\infty \right\} =$

$= t^2 e^{-t} = \dfrac{t^2}{e^t} \to 0,\ t\to +\infty$ Svar: 0

b) $\dfrac{d}{dx} x^{\arctan x}$:

Lösning: $\dfrac{d}{dx} x^{\arctan x} = \dfrac{d}{dx} e^{\arctan x \cdot \ln x} =$

$= e^{\arctan x \cdot \ln x}\left(\dfrac{1}{1+x^2}\cdot \ln x + \arctan x \cdot \dfrac{1}{x}\right) =$

$= x^{\arctan x} \cdot \dfrac{\ln x}{1+x^2} + \arctan x \cdot x^{\arctan x - 1}$

Svar: $x^{\arctan x} \cdot \dfrac{\ln x}{1+x^2} + \arctan x \cdot x^{\arctan x -1}$

② Rita grafen till $f(x) = \dfrac{x(x-3)}{x-4}$

Lösning: Vi ser att $D_f = \{x\in\mathbb{R} : x\neq 4\} = D_{f'}$.

Dessutom $f(x) = \dfrac{x\cdot(x-4)+x-4+4}{x-4} = x+1+\dfrac{4}{x-4}$

Härav följer att

$x=4$: vertikal asymptot då $\lim_{x\to 4\pm} f(x) = \pm\infty$

$y=x+1$ sned asymptot då $x\to\pm\infty$ eftersom

$\lim_{x\to\pm\infty}(f(x)-(x+1)) = 0$

Derivera $f(x)$.

$f'(x) = 1 - \dfrac{4}{(x-4)^2}$, $x\neq 4$.

Detta ger $f'(x)=0 \Leftrightarrow (x-4)^2-4=0 \Leftrightarrow (x-2)(x-6)=0$

Teckenstudium

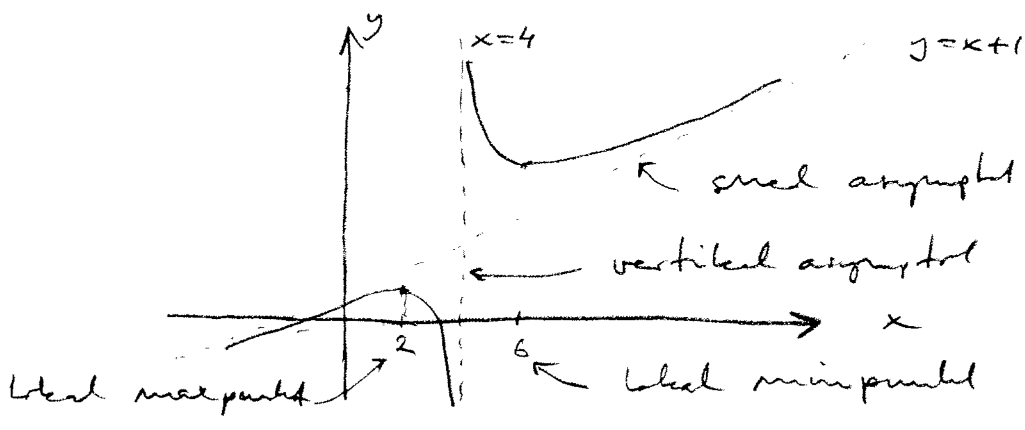
x		2		4		6	
$f'(x)$	+	0	−	∄	−	0	+
$f(x)$	↗		↘	∄	↘		↗

$f(0) = 0$

$f(2) = 1$ Lok. max

$f(6) = 9$ Lok. min

Rita grafen!



(3) a) $\lim_{x\to\infty}(1+\frac{1}{x})^{-x^3}\cdot e^{x^2-\frac{x}{2}}$

Lösning: Omskrivning ger

$$(1+\tfrac{1}{x})^{-x^3}\cdot e^{x^2-\frac{x}{2}} = e^{-x^3\ln(1+\frac{1}{x})+x^2-\frac{x}{2}}$$

Standardutveckling $\ln(1+t) = t-\frac{t^2}{2}+\frac{t^3}{3}+O(t^4)$, $t\to 0$

sätt $t=\frac{1}{x}$. Detta ger

$$\ln(1+\tfrac{1}{x}) = \tfrac{1}{x}-\tfrac{1}{2x^2}+\tfrac{1}{3x^3}+O(\tfrac{1}{x^4}),\quad x\to\infty$$

Alltså $-x^3\ln(1+\frac{1}{x})+x^2-\frac{x}{2} =$

$$= -x^3\left(\tfrac{1}{x}-\tfrac{1}{2x^2}+\tfrac{1}{3x^3}+O(\tfrac{1}{x^4})\right)+x^2-\tfrac{x}{2} =$$

$$= -\tfrac{1}{3}+O(\tfrac{1}{x}) \longrightarrow -\tfrac{1}{3} \text{ då } x\to\infty$$

Exponentialfunktionen är kontinuerlig så

$$e^{-\frac{1}{3}+O(\frac{1}{x})}\longrightarrow e^{-\frac{1}{3}},\quad x\to\infty$$

Svar: $e^{-\frac{1}{3}}$

b) Utveckla $e^{\sin x}\cdot\frac{e^{-x}}{\sqrt{1+x}}$ i potenser av x med restterm på formen $O(x^6)$

Lösning: Standardutvecklingar

$$\sin x = x-\frac{x^3}{6}+\frac{x^5}{120}+O(x^7)$$

$$\frac{1}{\sqrt{1+x}} = 1+(-\tfrac{1}{2})x+\frac{(-\frac{1}{2})(-\frac{1}{2}-1)}{2}x^2+\frac{(-\frac{1}{2})(-\frac{1}{2}-1)(-\frac{1}{2}-2)}{6}x^3+$$

$$+\frac{(-\frac{1}{2})(-\frac{1}{2}-1)(-\frac{1}{2}-2)(-\frac{1}{2}-3)}{24}x^4+\frac{(-\frac{1}{2})(-\frac{1}{2}-1)(-\frac{1}{2}-2)(-\frac{1}{2}-3)(-\frac{1}{2}-4)}{120}x^5+O(x^6)$$

$$= 1-\tfrac{1}{2}x+\tfrac{3}{8}x^2-\tfrac{5}{16}x^3+\tfrac{35}{128}x^4-\tfrac{63}{256}x^5+O(x^6)$$

$$e^t = 1+t+O(t^2)\qquad t\to 0$$

Insättning ger

$$e^{\sin x}\cdot\frac{e^{-x}}{\sqrt{1+x}} = e^{-\frac{x^3}{6}+\frac{x^5}{120}+O(x^7)}\cdot\left(1-\tfrac{1}{2}x+\tfrac{3}{8}x^2-\tfrac{5}{16}x^3+\right.$$

$$\left.+\tfrac{35}{128}x^4-\tfrac{63}{256}x^5+O(x^6)\right) = \left(1-\tfrac{x^3}{6}+\tfrac{x^5}{120}+O(x^6)\right)\cdot$$

$$\cdot\left(1-\tfrac{1}{2}x+\tfrac{3}{8}x^2-\tfrac{5}{16}x^3+\tfrac{35}{128}x^4-\tfrac{63}{256}x^5+O(x^6)\right) =$$

$$= 1-\tfrac{1}{2}x+\tfrac{3}{8}x^2+(-\tfrac{5}{16}-\tfrac{1}{6})x^3+(\tfrac{35}{128}-\tfrac{1}{6}\cdot(-\tfrac{1}{2}))x^4+$$

$$+(-\tfrac{63}{256}-\tfrac{1}{6}\cdot\tfrac{3}{8}+\tfrac{1}{120})x^5+O(x^6) =$$

$$= 1-\tfrac{1}{2}x+\tfrac{3}{8}x^2-\tfrac{23}{48}x^3+\tfrac{137}{384}x^4-\tfrac{1153}{3840}x^5+O(x^6).$$

Svar: se ovan

(4) a) Lös $z^2-2\sqrt{2}iz-2\sqrt{3}i=0$

Lösning: Kvadratkomplettering ger

$z^2 - 2\sqrt{2}iz - 2\sqrt{3}i = (z-\sqrt{2}i)^2 + 2 - 2\sqrt{3}i = 0$

Sätt $a+ib = z - \sqrt{2}i$, $a,b \in \mathbb{R}$

Vi får

$$\begin{cases} a^2 - b^2 + i2ab = -2 + 2\sqrt{3}i \\ a^2 + b^2 = \sqrt{(-2)^2 + (2\sqrt{3})^2} = 4 \end{cases}$$

Alltså $a^2 = 1$, $b^2 = 3$, $2ab = 2\sqrt{3}$ ger

$a+ib = 1 + i\sqrt{3}$ eller $-1 - i\sqrt{3}$

Detta ger $z = 1 + (\sqrt{3}+\sqrt{2})i$ eller $-1 + (\sqrt{2}-\sqrt{3})i$.

Svar: $1 + (\sqrt{2}+\sqrt{3})i$, $-1 + (\sqrt{2}-\sqrt{3})i$

b) Lös $\begin{cases} 4y'' + 4y' + y = 0 \\ y(0) = 2, \ y'(0) = 0 \end{cases}$

Lösning: Karakteristiska ekvationen: $4r^2 + 4r + 1 = 0$

har rötterna $r_1 = r_2 = -\frac{1}{2}$. Detta ger

$y(x) = Ae^{-\frac{1}{2}x} + Bxe^{-\frac{1}{2}x}$.

Begynnelsedata ger

$$\begin{cases} 2 = y(0) = A \\ 0 = y'(0) = -\frac{1}{2}A + B \end{cases} \quad \text{dvs} \quad \begin{cases} A = 2 \\ B = 1 \end{cases}$$

Insättning ger $y(x) = 2e^{-\frac{1}{2}x} + xe^{-\frac{1}{2}x}$

Svar: $2e^{-\frac{1}{2}x} + xe^{-\frac{1}{2}x}$

5 a) Samtliga primitiva funktioner till $\frac{x^3}{x-1}$

Lösning: Polynomdivision

$$\begin{array}{r|l} & x^2 + x + 1 \\ \hline x-1 & x^3 \\ & x^3 - x^2 \\ \hline & x^2 \\ & x^2 - x \\ \hline & x \\ & x - 1 \\ \hline & 1 \end{array}$$

ger $\frac{x^3}{x-1} = x^2 + x + 1 + \frac{1}{x-1}$

Integration ger $\int \frac{x^3}{x-1}\,dx = \frac{x^3}{3} + \frac{x^2}{2} + x + \ln|x-1| + C$

Svar: $\frac{x^3}{3} + \frac{x^2}{2} + x + \ln|x-1| + C$

b) Beräkna längden av kurvan $y = \ln(\sin x)$, $\frac{\pi}{4} \le x \le \frac{\pi}{2}$.

Lösning: Med x som parameter fås längden L

$L = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sqrt{1 + \left(\frac{\cos x}{\sin x}\right)^2}\,dx = \{ \sin x > 0 \text{ för } x \in [\frac{\pi}{4}, \frac{\pi}{2}] \} =$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{\sin x} \sqrt{\sin^2 x + \cos^2 x}\, dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{1-\cos^2 x} \cdot \sin x \, dx =$$

$$= \left\{ t = \cos x,\ dt = -\sin x\, dx,\ x = \tfrac{\pi}{4} \leftrightarrow t = \tfrac{1}{\sqrt{2}},\ x = \tfrac{\pi}{2} \leftrightarrow t = 0 \right\} =$$

$$= -\int_{\frac{1}{\sqrt{2}}}^{0} \frac{1}{1-t^2}\, dt = \int_{0}^{\frac{1}{\sqrt{2}}} \frac{1}{1-t^2}\, dt = \left\{ \frac{1}{1-t^2} = \frac{A}{1-t} + \frac{B}{1+t} = \right.$$

$$\left. = \frac{(1+t)A + (1-t)B}{1-t^2} = \frac{(A-B)t + A + B}{1-t^2},\quad A = \frac{1}{2} = B \right\} =$$

$$= \left[-\frac{1}{2}\ln|1-t| + \frac{1}{2}\ln|1+t| \right]_0^{\frac{1}{\sqrt{2}}} = \frac{1}{2}\ln\left(\frac{1+\frac{1}{\sqrt{2}}}{1-\frac{1}{\sqrt{2}}}\right) = \ldots =$$

$$= \frac{1}{2}\ln\left((1+\sqrt{2})^2\right) = \ln(1+\sqrt{2})$$

Svar: $\ln(1+\sqrt{2})$.

lite alternativ lösning till 3b,

$$e^{\sin x}\cdot\frac{e^{-x}}{\sqrt{1+x}} = e^{\sin x - x - \frac{1}{2}\ln(1+x)}$$

Med standardutvecklingarna

$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} + \mathcal{O}(x^7)$$

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 + \mathcal{O}(x^6)$$

fås $\sin x - x - \frac{1}{2}\ln(1+x) = -\frac{x^3}{6} + \frac{x^5}{120} - \frac{1}{2}\left(x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5\right) +$

$$+\mathcal{O}(x^6) = -\frac{1}{2}x + \frac{1}{4}x^2 - \frac{1}{3}x^3 + \frac{1}{8}x^4 - \frac{11}{120}x^5 + \mathcal{O}(x^6).$$

Insättning i standardutvecklingen $e^t = 1 + t + \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{24} + \frac{t^5}{120} + \mathcal{O}(t^6)$

ger

$$e^{\sin x}\cdot\frac{e^{-x}}{\sqrt{1+x}} = 1 + \left(-\frac{1}{2}x + \frac{1}{4}x^2 - \frac{1}{3}x^3 + \frac{1}{8}x^4 - \frac{11}{120}x^5\right) +$$

$$+ \frac{1}{2}\Big(\frac{1}{4}x^2 + \frac{1}{16}x^4 + 2\cdot\left(-\frac{1}{2}\right)\frac{1}{4}x^3 + 2\left(-\frac{1}{2}\right)\left(-\frac{1}{3}\right)x^4 + 2\left(-\frac{1}{2}\right)\left(\frac{1}{8}\right)x^5 +$$

$$+ 2\left(\frac{1}{4}\right)\left(-\frac{1}{3}\right)x^5\Big) + \frac{1}{6}\Big(-\frac{1}{8}x^3 + 3\left(-\frac{1}{2}\right)^2\left(\frac{1}{4}\right)x^4 + 3\left(-\frac{1}{2}\right)^2\left(-\frac{1}{3}\right)x^5 + 3\left(-\frac{1}{2}\right)\left(\frac{1}{4}\right)^2 x^5\Big) +$$

$$+ \frac{1}{24}\left(\frac{1}{16}x^4 + 4\left(-\frac{1}{2}\right)^3\cdot\left(\frac{1}{4}\right)x^5\right) + \frac{1}{120}\left(-\frac{1}{32}x^5\right) + \mathcal{O}(x^6) =$$

$$= 1 - \frac{1}{2}x + \left(\frac{1}{4} + \frac{1}{8}\right)x^2 + \left(-\frac{1}{3} - \frac{1}{8} - \frac{1}{48}\right)x^3 +$$

$$+ \left(\frac{1}{8} + \frac{1}{32} + \frac{1}{6} + \frac{1}{32} + \frac{1}{24\cdot 16}\right)x^4 + \left(-\frac{11}{120} - \frac{1}{16} - \frac{1}{12} - \frac{1}{24} - \frac{1}{64} -\right.$$

$$\left. - \frac{1}{24\cdot 8} - \frac{1}{120\cdot 32}\right)x^5 + \mathcal{O}(x^6) =$$

$$= 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{23}{48}x^3 + \frac{137}{384}x^4 - \frac{1153}{3840}x^5 + \mathcal{O}(x^6).$$