# Solutions to exercises (week 3)

## Exercise 5, Ref. [1] Sec. 2.2

Suppose $u > r > 0 \geq d$. A non-standard European derivative with maturity time $N$ has pay-off $Y = S(N)$ if $S(0) < S(1) < \cdots < S(N)$ and $Y = S(0)$ otherwise. Find $\Pi_Y(0)$.

### Solution

We have the general formula

$$\Pi_Y(0) = e^{-rN} \sum_{x \in \{u,d\}^N} (q_u)^{N_u(x)} (q_d)^{N_d(x)} Y(x), \tag{1}$$

where $(q_u, q_d)$ is the risk neutral probability, $N_u(x)$ is the number of "$u$" in the path $x$ and $N_d(x) = N - N_u(x)$ is the number of "$d$" in the path $x$. Moreover $Y(x)$ denotes the pay-off as a function of the path of the stock price. The exercise tells us that

$$Y(x) = S(N, x), \quad \text{if } x = x_* = (u, u, \dots u), \text{ i.e., } x_i = u \text{ for all } i = 1, \dots N,$$

while $Y(x) = S(0)$ for $x \neq x_*$. Moreover, since $S(N, x_*) = S(0)e^{Nu}$, then

$$Y(x_*) = S(0)e^{Nu}.$$

Since in addition $N_u(x_*) = N$, we can rewrite the sum (1) as

$$\begin{aligned}
\Pi_Y(0) &= e^{-rN} (q_u)^{N_u(x_*)} (q_d)^{N_d(x_*)} Y(x_*) + e^{-rN} \sum_{x \neq x_*} (q_u)^{N_u(x)} (q_d)^{N_d(x)} Y(x) \\
&= e^{-rN} (q_u)^N S(0) e^{Nu} + e^{-rN} \sum_{x \neq x_*} (q_u)^{N_u(x)} (q_d)^{N_d(x)} Y(x). \qquad (2)
\end{aligned}$$

Next we compute the sum on $x \neq x_*$. First replacing $N_d(x) = N - N_u(x)$ and $Y(x) = S(0)$ we have

$$\sum_{x \neq x_*} (q_u)^{N_u(x)} (q_d)^{N_d(x)} Y(x) = S(0)(q_d)^N \sum_{x \neq x_*} \left( \frac{q_u}{q_d} \right)^{N_u(x)}. \tag{3}$$

Now, $N_u(x)$ takes value in $\{0, 1, \dots, N-1\}$; it cannot be equal to $N$ because the only element in $\{u, d\}^N$ for which $N_u(x) = N$ is $x_*$, but this element is not taken into account in the sum that we are computing. Using that the number of $x \in \{u, d\}^N$ for which $N_u(x) = k$ is given by the binomial coefficient $\binom{N}{k}$, we obtain

$$\sum_{x \neq x_*} \left(\frac{q_u}{q_d}\right)^{N_u(x)} = \sum_{k=0}^{N-1} \binom{N}{k} \left(\frac{q_u}{q_d}\right)^k .$$

Adding and subtracting the term $k = N$ (where we use that $\binom{N}{N} = 1$), we find

$$\begin{aligned}\sum_{x \neq x_*} \left(\frac{q_u}{q_d}\right)^{N_u(x)} &= \sum_{k=0}^{N-1} \binom{N}{k} \left(\frac{q_u}{q_d}\right)^k \\ &= \sum_{k=0}^{N} \binom{N}{k} \left(\frac{q_u}{q_d}\right)^k - \left(\frac{q_u}{q_d}\right)^N \\ &= \left(1 + \frac{q_u}{q_d}\right)^N - \left(\frac{q_u}{q_d}\right)^N ,\end{aligned}$$

where for the last equality we use the binomial theorem: $(1+a)^N = \sum_{k=0}^{N} \binom{N}{k} a^k$. Substituting into (3) and using that $q_u + q_d = 1$ we obtain

$$\sum_{x \neq x_*} (q_u)^{N_u(x)} (q_d)^{N_d(x)} Y(x) = S(0)\big(1 - (q_u)^N\big).$$

Finally, replacing in (2) we find

$$\Pi_Y(0) = e^{-rN} S(0) \left[(q_u)^N e^{Nu} + 1 - (q_u)^N\right] .$$

# Exercise 3.1 Ref. [1]

Let the price of a stock $S(t)$ be given by the $N$-period binomial model with parameters $u, d, p$ and let $B(t) = B_0 e^{rt}$ be the value of the risk-free asset, where $d < r < u$. Let $C(t, S(t), K, N)$ and $P(t, S(t), K, N)$ be the binomial price of the European call and European put with strike $K$ and maturity $N$. Show that these functions satisfy the properties in Theorem 1.1, namely:

1 The **put-call parity** holds

$$S(t) - C(t, S(t), K, N) = Ke^{-r(N-t)} - P(t, S(t), K, N). \tag{4}$$

2 If $r \geq 0$, then $C(t, S(t), K, N) \geq (S(t) - K)_+$; the strict inequality $C(t, S(t), K, N) > (S(t) - K)_+$ holds when $r > 0$.

3 If $r \geq 0$, the map $N \to C(t, S(t), K, N)$ is non-decreasing.

4 The maps $K \to C(t, S(t), K, N)$ and $K \to P(t, S(t), K, N)$ are convex[1].

[1]Recall that a real-valued function $f$ on an interval $I$ is convex if $f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta)f(y)$, for all $x, y \in I$ and $\theta \in (0, 1)$.

## Solution

Recall that the binomial price of a European derivative with pay-off $Y$ and maturity $N$ is

$$\Pi_Y(t) := e^{-r(N-t)} \sum_{(x_{t+1},\dots x_N)\in\{u,d\}^{N-t}} q_{x_{t+1}} \cdots q_{x_N} Y(x), \quad x = (x_1, \dots, x_N).$$

For the European call we have $Y(x) = (S(N,x) - K)_+ = (S(t)e^{x_{t+1}+\dots+x_N} - K)_+$, hence

$$C(t, S(t), K, N) = e^{-r(N-t)} \sum_{(x_{t+1},\dots x_N)\in\{u,d\}^{N-t}} q_{x_{t+1}} \cdots q_{x_N} (S(t)e^{x_{t+1}+\dots+x_N} - K)_+.$$

Similarly for the European put we obtain

$$P(t, S(t), K, N) = e^{-r(N-t)} \sum_{(x_{t+1},\dots x_N)\in\{u,d\}^{N-t}} q_{x_{t+1}} \cdots q_{x_N} (K - S(t)e^{x_{t+1}+\dots+x_N})_+.$$

Now let $k$ be the number of $u$ in $(x_{t+1}, \dots, x_N)$ and $N - t - k$ be the number of $d$. Then

$$e^{x_{t+1}+\dots+x_N} = e^{ku+(N-t-k)d}, \quad q_{x_{t+1}} \cdots q_{x_N} = q_u^k q_d^{N-t-k}.$$

Now, the number of paths $(x_{t+1}, \dots, x_N)$ for which the number of $u$ is $k$ is given by the binomial coefficient $\binom{N-t}{k}$. Hence we can rewrite the definition of $C(t, S(t), K, N)$ and $P(t, S(t), K, N)$ as

$$C(t, S(t), K, N) = e^{-r(N-t)} \sum_{k=0}^{N-t} \binom{N-t}{k} q_u^k q_d^{N-t-k} (S(t)e^{ku+(N-t-k)d} - K)_+$$

$$P(t, S(t), K, N) = e^{-r(N-t)} \sum_{k=0}^{N-t} \binom{N-t}{k} q_u^k q_d^{N-t-k} (K - S(t)e^{ku+(N-t-k)d})_+$$

We can now prove the properties 1-4.

1 We have

$$\begin{aligned} C(t, S(t), K, N) - P(t, S(t), K, N) =& e^{-r(N-t)} \sum_{k=0}^{N-t} \binom{N-t}{k} q_u^k q_d^{N-t-k} \\ & \times [(S(t)e^{ku+(N-t-k)d} - K)_+ - (K - S(t)e^{ku+(N-t-k)d})_+] \end{aligned}$$

Now we use that $(z - K)_+ - (K - z)_+ = z - K$, for all $z$, hence

$$\begin{aligned} C - P &= e^{-r(N-t)} \sum_{k=0}^{N-t} \binom{N-t}{k} q_u^k q_d^{N-t-k} (S(t)e^{ku+(N-t-k)d} - K) \\ &= S(t)e^{-r(N-t)} \sum_{k=0}^{N-t} \binom{N-t}{k} q_u^k q_d^{N-t-k} e^{ku+(N-t-k)d} \\ &\quad - K \sum_{k=0}^{N-t} \binom{N-t}{k} q_u^k q_d^{N-t-k} = S(t)I_1 - KI_2. \end{aligned}$$

Hence the put-call parity follows if we show that $I_2 = e^{-r(N-t)}$ and $I_1 = 1$. We have

$$\begin{aligned} I_1 &= e^{-r(N-t)} \sum_{k=0}^{N-t} \binom{N-t}{k} q_u^k q_d^{N-t-k} e^{ku+(N-t-k)d} \\ &= e^{-r(N-t)} (q_d e^d)^{N-t} \sum_{k=0}^{N-t} \binom{N-t}{k} \left(\frac{q_u e^u}{q_d e^d}\right)^k \end{aligned}$$

Using the binomial theorem $(1+a)^N = \sum_{k=0}^{N} \binom{N}{k} a^k$ and the identity $q_u e^u + q_d e^d = e^r$ we obtain

$$\begin{aligned} I_1 &= e^{-r(N-t)} (q_d e^d)^{N-t} \left(1 + \frac{q_u e^u}{q_d e^d}\right)^{N-t} \\ &= e^{-r(N-t)} (q_u e^u + q_d e^d)^{N-t} = 1. \end{aligned}$$

The proof that $I_2 = e^{-r(N-t)}$ is similar.

2 The proof follows by the put-call parity as in Theorem 1.1

3 We want to show that

$$C(t, S(t), K, N) \leq C(t, S(t), K, N+1)$$

Note that for $r = 0$ the claim is obvious. The case $r > 0$ is quite technical. Let us prove it. In the definition of $C(N+1) = C(t, S(t), K, N+1)$ we replace the *Pascal identity*

$$\binom{N+1-t}{k} = \binom{N-t}{k-1} + \binom{N-t}{k}$$

(with the convention $\binom{N}{-1} = 0$) and obtain

$$\begin{aligned} C(N+1) =& e^{-r(N+1-t)} \Bigg\{ \sum_{k=1}^{N+1-t} \binom{N-t}{k-1} q_u^k q_d^{N+1-t-k} (S(t) e^{ku+(N+1-t-k)d} - K)_+ \\ &+ \sum_{k=0}^{N+1-t} \binom{N-t}{k} q_u^k q_d^{N+1-t-k} (S(t) e^{ku+(N+1-t-k)d} - K)_+ \Bigg\}. \end{aligned}$$

In the first sum we make the change of index $j = k-1$, while for the second sum we use that it is greater than the sum extended only up to $N-t$ (i.e., we neglect the last term $k = N+1-t$). So doing we obtain

$$\begin{aligned}
C(N+1) \geq & e^{-r(N+1-t)} \left\{ \sum_{j=0}^{N-t} \binom{N-t}{j} q_u^{j+1} q_d^{N-t-j} (S(t)e^{(j+1)u+(N-t-j)d} - K)_+ \right. \\
& \left. + \sum_{k=0}^{N-t} \binom{N-t}{k} q_u^k q_d^{N+1-t-k} (S(t)e^{ku+(N+1-t-k)d} - K)_+ \right\} \\
& = e^{-r(N+1-t)} \left\{ \sum_{j=0}^{N-t} \binom{N-t}{j} q_u^j q_d^{N-t-j} q_u (S(t)e^{ju+(N-t-j)d} e^u - K)_+ \right. \\
& \left. + \sum_{k=0}^{N-t} \binom{N-t}{k} q_u^k q_d^{N-t-k} q_d (S(t)e^{ku+(N-t-k)d} e^d - K)_+ \right\} \\
& = e^{-r(N+1-t)} \sum_{k=0}^{N-t} \binom{N-t}{k} q_u^k q_d^{N-t-k} \\
& \times [(S(t)e^{ku+(N-t-k)d} q_u e^u - Kq_u)_+ + (S(t)e^{ku+(N-t-k)d} q_d e^d - Kq_d)_+
\end{aligned}$$

Using the simple inequality $(y)_+ + (z)_+ \geq (y+z)_+$, we obtain

$$\begin{aligned}
C(N+1) \geq e^{-r(N+1-t)} & \sum_{k=0}^{N-t} \binom{N-t}{k} q_u^k q_d^{N-t-k} \\
& \times [(S(t)e^{ku+(N-t-k)d} (q_u e^u + q_d e^d) - K(q_u + q_d))_+
\end{aligned}$$

As $q_u e^u + q_d e^d = e^r$, $q_u + q_d = 1$ and $r \geq 0$ we find

$$\begin{aligned}
C(N+1) & \geq e^{-r(N+1-t)} \sum_{k=0}^{N-t} \binom{N-t}{k} q_u^k q_d^{N-t-k} (S(t)e^{ku+(N-t-k)d} e^r - K)_+ \\
& = e^{-r(N-t)} \sum_{k=0}^{N-t} \binom{N-t}{k} q_u^k q_d^{N-t-k} (S(t)e^{ku+(N-t-k)d} - Ke^{-r})_+ \\
& \geq C(N).
\end{aligned}$$

4 The only dependence on $K$ of the functions $C(t, S(t), K, N)$, $P(t, S(t), K, N)$ is through the terms $(z-K)_+$, $(K-z)_+$. As both these functions are convex in $K$ (draw a picture), the result follows.

# Exercise 3.2 Ref. [1]

Consider a 3-period binomial asset pricing model with the following parameters:

$$e^u = \frac{5}{4}, \quad e^d = \frac{1}{2}, \quad e^r = 1 \quad p = \frac{1}{2}.$$

Assume $S(0) = \frac{64}{25}$. Consider a European derivative expiring at time $T = 2$ and with pay-off

$$Y = S(3)H(S(3) - 1),$$

where $H$ is the Heaviside function: $H(x) = 0$, if $x < 0$, $H(x) = 1$ if $x \geq 0$ (this is an example of a so called **digital option**). Compute the possible paths of the derivative price and for each of them give the number of shares of the underlying stock in the hedging portfolio process. Compute the probability that the return of a constant portfolio with a short position in the derivative be positive.

## Solution

We start by writing down the diagram of the stock price and the value of the derivative at time of maturity $T = 3$ (which is equal to the pay-off)

$$\begin{tikzcd}
 & & & S(3)=5 \Rightarrow \Pi_Y(3)=5 \\
 & & S(2)=4 \arrow[ur, "u"] \arrow[dr, "d"] & \\
 & S(1)=\frac{16}{5} \arrow[ur, "u"] \arrow[dr, "d"] & & S(3)=2 \Rightarrow \Pi_Y(3)=2 \\
S(0)=\frac{64}{25} \arrow[ur, "u"] \arrow[dr, "d"] & & S(2)=\frac{8}{5} \arrow[ur, "u"] \arrow[dr, "d"] & \\
 & S(1)=\frac{32}{25} \arrow[ur, "u"] \arrow[dr, "d"] & & S(3)=\frac{4}{5} \Rightarrow \Pi_Y(3)=0 \\
 & & S(2)=\frac{16}{25} \arrow[ur, "u"] \arrow[dr, "d"] & \\
 & & & S(3)=\frac{8}{25} \Rightarrow \Pi_Y(3)=0
\end{tikzcd}$$

The parameters of the binomial model are such that

$$q_u = \frac{2}{3}, \quad q_d = \frac{1}{3}, \quad r = 0.$$

To compute the price of the derivative at the times $t \in \{0, 1, 2\}$ we use the recurrence formula

$$\Pi_Y(t) = e^{-r}(q_u \Pi_Y^u(t+1) + q_d \Pi_Y^d(t+1)) = \frac{2}{3}\Pi_Y^u(t+1) + \frac{1}{3}\Pi_Y^d(t+1), \quad t \in \{0, 1, 2\}.$$

Hence at time $t = 2$ we have

$$
\begin{aligned}
S(2) = 4 &\Rightarrow \Pi_Y(2) = \frac{2}{3} \cdot 5 + \frac{1}{3} \cdot 2 = 4 \\
S(2) = \frac{8}{5} &\Rightarrow \Pi_Y(2) = \frac{2}{3} \cdot 2 + \frac{1}{3} \cdot 0 = \frac{4}{3} \\
S(2) = \frac{16}{25} &\Rightarrow \Pi_Y(2) = \frac{2}{3} \cdot 0 + \frac{1}{3} \cdot 0 = 0.
\end{aligned}
$$

At time $t = 1$ we have

$$
\begin{aligned}
S(1) = \frac{16}{5} &\Rightarrow \Pi_Y(1) = \frac{2}{3} \cdot 4 + \frac{1}{3} \cdot \frac{4}{3} = \frac{28}{9} \\
S(1) = \frac{32}{25} &\Rightarrow \Pi_Y(1) = \frac{2}{3} \cdot \frac{4}{3} + \frac{1}{3} \cdot 0 = \frac{8}{9}
\end{aligned}
$$

and at time $t = 0$ we have

$$\Pi_Y(0) = \frac{2}{3} \cdot \frac{28}{9} + \frac{1}{3} \cdot \frac{8}{9} = \frac{64}{27}$$

Hence we obtain the following diagram for the derivative price

$$
\begin{tikzcd}
 & & & \Pi_Y(3) = 5 \\
 & & \Pi_Y(2) = 4 \arrow[ur, "u"] \arrow[dr, "d"] & \\
 & \Pi_Y(1) = \frac{28}{9} \arrow[ur, "u"] \arrow[dr, "d"] & & \Pi_Y(3) = 2 \\
\Pi_Y(0) = \frac{64}{27} \arrow[ur, "u"] \arrow[dr, "d"] & & \Pi_Y(2) = \frac{4}{3} \arrow[ur, "u"] \arrow[dr, "d"] & \\
 & \Pi_Y(1) = \frac{8}{9} \arrow[ur, "u"] \arrow[dr, "d"] & & \Pi_Y(3) = 0 \\
 & & \Pi_Y(2) = 0 \arrow[ur, "u"] \arrow[dr, "d"] & \\
 & & & \Pi_Y(3) = 0
\end{tikzcd}
$$

This concludes the first part of the exercise. To compute the number of shares of the underlying asset in the hedging portfolio we use the formula

$$h_S(t+1) = \frac{1}{S(t)} \frac{\Pi_Y^u(t+1) - \Pi_Y^d(t+1)}{e^u - e^d}$$

for $t = 1, 2$ and $h_S(0) = h_S(1)$, where we recall that $h_S(t+1)$ is the position in the interval $(t, t+1]$. Letting $t = 2$ we obtain

$$h_S(3) = \frac{4}{3}\frac{\Pi_Y^u(3) - \Pi_Y^d(3)}{S(2)},$$

whence

$$\begin{aligned}
S(2) = 4 &\Rightarrow h_S(3) = \frac{4}{3} \cdot \frac{5-2}{4} = 1 \\
S(2) = \frac{8}{5} &\Rightarrow h_S(3) = \frac{4}{3} \cdot \frac{2-0}{8/5} = \frac{5}{3} \\
S(2) = \frac{16}{25} &\Rightarrow h_S(3) = 0.
\end{aligned}$$

Likewise

$$h_S(2) = \frac{4}{3}\frac{\Pi_Y^u(2) - \Pi_Y^d(2)}{S(1)}.$$

Hence

$$\begin{aligned}
S(1) = \frac{16}{25} &\Rightarrow h_S(2) = \frac{4}{3} \cdot \frac{4 - 4/3}{16/5} = \frac{10}{9} \\
S(1) = \frac{32}{25} &\Rightarrow h_S(2) = \frac{4}{3} \cdot \frac{4/3 - 0}{32/25} = \frac{25}{18}
\end{aligned}$$

and finally

$$h_S(0) = h_S(1) = \frac{4}{3}\frac{\Pi_Y^u(1) - \Pi_Y^d(1)}{S(0)} = \frac{4}{3} \cdot \frac{28/9 - 8/9}{64/25} = \frac{125}{108}.$$

This concludes the second part of the exercise. Consider now a constant portfolio with -1 shares of the derivative. The return of this portfolio is positive if the value of the derivative at the expiration date is smaller than the initial value. This happens along all paths except $x = (u, u, u)$, hence the probability that the return of this portfolio be positive is $1 - (p_u)^3 = 1 - 1/8 = 7/8 = 87.5\%$.

# Exercise 3.3, Ref. [1]

Consider a standard European derivative with pay-off $Y = g(S(2))$ at the time of maturity 2. Assume that the price of the underlying stock follows the 2-period arbitrage-free binomial model

$$S(t) = \begin{cases} S(t-1)e^u & \text{with probability } p, \\ S(t-1)e^d & \text{with probability } (1-p) \end{cases} \qquad t = 1, 2$$

and that the interest rate of the risk-free asset is a constant $r > 0$. Let

$$\Delta = g(S_0 e^{2d}) - e^{d-u} g(S_0 e^{2d}) - g(S_0 e^{u+d}) + g(S_0 e^{2u}) e^{d-u}.$$

Show that a constant predictable hedging portfolio $(h_S, h_B)$ exists if and only if $\Delta = 0$ and find such portfolio.

## Solution

The hedging condition reads

$$h_S S(2) + h_B B_0 e^{2r} = g(S(2)).$$

Since the portfolio is constant and is required to be predictable, then it can only depend on $S_0 = S(0)$ and not on $S(1), S(2)$. Hence we have to express $S(2)$ in terms of $S_0$ in the previous equation. Since $S(2) \in \{S(0)e^{2u}, S_0 e^{u+d}, S_0 e^{2d}\}$, we obtain the system

$$\begin{aligned}
h_S S_0 e^{2u} + h_B B_0 e^{2r} &= g(S_0 e^{2u}) \\
h_S S_0 e^{u+d} + h_B B_0 e^{2r} &= g(S_0 e^{u+d}) \\
h_S S_0 e^{2d} + h_B B_0 e^{2r} &= g(S_0 e^{2d}).
\end{aligned}$$

It is straightforward to show that the previous system has a (unique) solution $(h_S, h_B)$ if and only if $\Delta = 0$ and in this case the solution is given by

$$h_B = \frac{e^u g(S_0 e^{u+d}) - g(S_0 e^{2u}) e^d}{B_0 e^{2r}(e^d - e^u)}, \quad h_S = \frac{g(S_0 e^{2u})(2e^d - e^u) - e^u g(S_0 e^{u+d})}{S_0 e^{2u}(e^d - e^u)}.$$