Solutions to selected exercises (week 4)

Exercise 4.3, Ref. [1]

Show that the binomial price of a standard American derivative at time t is a deterministic function of $S(t)$. Moreover show that the binomial price of a standard American derivative is always greater than or equal to the binomial price of the corresponding European derivative. Finally show that an American call and a European call with the same strike and maturity have the same binomial price (assuming $r \geq 0$[1]).

Solution

Recall that the binomial price of a standard American derivative with intrinsic value $Y(t) = g(S(t))$ is given by

$$\widehat{\Pi}_Y(N) = Y(N) \tag{1}$$

$$\widehat{\Pi}_Y(t) = \max(Y(t), e^{-r}(q_u \widehat{\Pi}_Y^u(t+1) + q_d \widehat{\Pi}_Y^d(t+1)), \quad t \in \{0, \dots, N-1\}, \tag{2}$$

see Definition 4.2, Ref. [1]. To prove that $\widehat{\Pi}_Y(t)$ is a deterministic function of $S(t)$, i.e., $\widehat{\Pi}_Y(t) = H_t(S(t))$ for some functions $H_t : (0, \infty) \to \mathbb{R}$, we can argue by induction. First this is true at time N, because $\widehat{\Pi}_Y(N) = Y(N) = g(S(N))$, hence $H_N = g$. Now assume that the claim is true at time $t+1$, i.e., there exists $H_{t+1} : (0, \infty) \to \mathbb{R}$ such that $\widehat{\Pi}_Y(t+1) = H_{t+1}(S(t+1))$. Hence

$$\widehat{\Pi}_Y^u(t+1) = H_{t+1}(S(t)e^u), \quad \widehat{\Pi}_Y^d(t+1) = H_{t+1}(S(t)e^d).$$

Thus, using $Y(t) = g(S(t))$, we have

$$\widehat{\Pi}_Y(t) = \max(g(S(t)), e^{-r}(q_u H_{t+1}(S(t)e^u) + q_d H_{t+1}(S(t)e^d)) = H_t(S(t)),$$

where $H_t(z) = \max(g(z), e^{-r}(q_u H_{t+1}(ze^u) + q_d H_{t+1}(ze^d))$. This concludes the first part of the exercise.

To prove that the price of the American derivative is greater or equal to the the price of the corresponding European derivative, i.e., $\widehat{\Pi}_Y(t) \geq \Pi_Y(t)$, we can also argue by induction.

[1] The information $r \geq 0$ is missing in the text of the exercise in the lectures notes!

The claim is true at maturity N, as at this time the value of both derivatives equals the pay-off by definition. Now assume the claim is true at time $t+1$, i.e.,

$$\widehat{\Pi}_Y(t+1) \geq \Pi_Y(t+1).$$

Hence

$$e^{-r}(q_u\widehat{\Pi}^u_Y(t+1) + q_d\widehat{\Pi}^d_Y(t+1)) \geq e^{-r}(q_u\Pi^u_Y(t+1) + q_d\Pi^d_Y(t+1)),$$

which implies

$$\begin{aligned}\widehat{\Pi}_Y(t) &= \max(Y(t), e^{-r}(q_u\widehat{\Pi}^u_Y(t+1) + q_d\widehat{\Pi}^d_Y(t+1)) \geq e^{-r}(q_u\widehat{\Pi}^u_Y(t+1) + q_d\widehat{\Pi}^d_Y(t+1)) \\ &\geq e^{-r}(q_u\Pi^u_Y(t+1) + q_d\Pi^d_Y(t+1)) = \Pi_Y(t),\end{aligned}$$

where we used that the price of European calls satisfies the recurrence relation

$$\Pi_Y(t) = e^{-r}(q_u\Pi^u_Y(t+1) + q_d\Pi^d_Y(t+1)), \tag{3}$$

see Theorem 3.1, Ref. [1].

To prove that American and European calls with the same parameters have the same value, we first observe that it suffices to prove that $\widehat{\Pi}_{call}(t) \leq \Pi_{call}(t)$, since we already proved that $\widehat{\Pi}_{call}(t) \geq \Pi_{call}(t)$ holds (we prove this for all standard American derivatives!). We argue again by induction. At maturity $\widehat{\Pi}_{call}(N) = \Pi_{call}(N)$. Assume that $\widehat{\Pi}_{call}(t+1) \leq \Pi_{call}(t+1)$. Then

$$e^{-r}(q_u\widehat{\Pi}^u_{call}(t+1) + q_d\widehat{\Pi}^d_{call}(t+1)) \leq e^{-r}(q_u\Pi^u_{call}(t+1) + q_d\Pi^d_{call}(t+1)). \tag{4}$$

Now let $\Pi_{put}(t)$ be the price of the European put with the same parameters as the European call. We have shown in Exercise 3.1 that these values satisfy the put-call parity:

$$\Pi_{call}(t) = S(t) - Ke^{-r(T-t)} + \Pi_{put}(t).$$

As $\Pi_{put}(t) \geq 0$ and $r \geq 0$ we have $\Pi_{call}(t) \geq S(t) - K$. Since $\Pi_{call}(t) \geq 0$, we have

$$\Pi_{call}(t) \geq \max(S(t) - K, 0) = (S(t) - K)_+ = Y(t),$$

where $Y(t)$ is the intrinsic value of the American call. As

$$\Pi_{call}(t) = e^{-r}(q_u\Pi^u_{call}(t+1) + q_d\Pi^d_{call}(t+1)),$$

we have

$$Y(t) \leq e^{-r}(q_u\Pi^u_{call}(t+1) + q_d\Pi^d_{call}(t+1)).$$

Hence, using that $\max(a, b) \leq \max(c, b)$ holds when $a \leq c$,

$$\begin{aligned}\widehat{\Pi}_{call}(t) &= \max(Y(t), e^{-r}(q_u\widehat{\Pi}^u_{call}(t+1) + q_d\widehat{\Pi}^d_{call}(t+1)) \\ &\leq \max(e^{-r}(q_u\Pi^u_{call}(t+1) + q_d\Pi^d_{call}(t+1)), e^{-r}(q_u\widehat{\Pi}^u_{call}(t+1) + q_d\widehat{\Pi}^d_{call}(t+1)))\end{aligned}$$

By (4),

$$\begin{aligned}&\max(e^{-r}(q_u\Pi^u_{call}(t+1) + q_d\Pi^d_{call}(t+1)), e^{-r}(q_u\widehat{\Pi}^u_Y(t+1) + q_d\widehat{\Pi}^d_Y(t+1))) \\ &\quad = e^{-r}(q_u\Pi^u_{call}(t+1) + q_d\Pi^d_{call}(t+1)) = \Pi_{call}(t).\end{aligned}$$

Hence $\widehat{\Pi}_{call}(t) \leq \Pi_{call}Y(t)$, and the proof is complete.

Exercise 4.6 Ref. [1]

Consider a 2-period binomial model with the following parameters:

$$e^u = \frac{7}{4}, \quad e^d = \frac{1}{2}, \quad e^r = \frac{9}{8}.$$

Let $S(0) = 1$ be the initial price of the stock and consider an American put with strike $K = \frac{3}{4}$ at maturity time $N = 2$. Find the hedging portfolio for this derivative.

Solution

The binomial tree for the stock price is

$$\begin{tikzcd}
 & & S(2) = \frac{49}{16} \\
 & S(1) = \frac{7}{4} \arrow[ur, "u"] \arrow[dr, "d"] & \\
S_0 = 1 \arrow[ur, "u"] \arrow[dr, "d"] & & S(2) = \frac{7}{8} \\
 & S(1) = \frac{1}{2} \arrow[ur, "u"] \arrow[dr, "d"] & \\
 & & S(2) = \frac{1}{4}
\end{tikzcd}$$

The binomial tree for the price of the American derivative (computed in the class) is

$$\begin{tikzcd}
 & & \widehat{\Pi}_Y(2) = 0 \\
 & \widehat{\Pi}_Y(1) = 0 \arrow[ur, "u"] \arrow[dr, "d"] & \\
\widehat{\Pi}_Y(0) = \frac{1}{9} \arrow[ur, "u"] \arrow[dr, "d"] & & \widehat{\Pi}_Y(2) = 0 \\
 & \widehat{\Pi}_Y(1) = \frac{1}{4} \arrow[ur, "u"] \arrow[dr, "d"] & \\
 & & \widehat{\Pi}_Y(2) = \frac{1}{2}
\end{tikzcd}$$

To compute the hedging portfolio $\{\widehat{h}_S(t), \widehat{h}_B(t)\}_{t=1,2}$ we use the formula

$$\widehat{h}_S(0) = \widehat{h}_S(1), \quad \widehat{h}_B(0) = \widehat{h}_B(1),$$

$$\widehat{h}_S(1) = \frac{1}{S(0)} \frac{\widehat{\Pi}^u_Y(1) - \widehat{\Pi}^d_Y(1)}{e^u - e^d}, \quad \widehat{h}_B(1) = \frac{e^{-r}}{B(0)} \frac{e^u \widehat{\Pi}^d_Y(1) - e^d \widehat{\Pi}^u_Y(1)}{e^u - e^d}. \tag{5}$$

$$\widehat{h}_S(2) = \frac{1}{S(1)} \frac{\widehat{\Pi}^u_Y(2) - \widehat{\Pi}^d_Y(2)}{e^u - e^d}, \quad \widehat{h}_B(2) = \frac{e^{-r}}{B(1)} \frac{e^u \widehat{\Pi}^d_Y(2) - e^d \widehat{\Pi}^u_Y(2)}{e^u - e^d}. \tag{6}$$

see Theorem 4.1, Ref. [1]. Since $\widehat{\Pi}_Y(2) = Y$, we have

$$\widehat{h}_S(2) = \widehat{h}_B(2) = 0, \quad \text{if } S(1) = \frac{7}{4},$$

which is obvious since the American put is worthless when the price of the stock goes up at time 1 and thus we don't need to hedge it! However when $S(1) = 1/2$ we obtain

$$\widehat{h}_S(2) = \frac{1}{\frac{1}{2}} \frac{0 - \frac{1}{2}}{\frac{7}{4} - \frac{1}{2}} = -\frac{4}{5},$$

$$\widehat{h}_B(2) = \frac{\frac{8}{9}}{B(1)} \frac{\frac{7}{4}\frac{1}{2} - \frac{1}{2}0}{\frac{7}{4} - \frac{1}{2}} = \frac{1}{B_0}\left(\frac{8}{9}\right)^2 \frac{7}{10}$$

The position at time 1 (and 0) is computed likewise:

$$\widehat{h}_S(1) = \frac{1}{1} \frac{0 - \frac{1}{4}}{\frac{7}{4} - \frac{1}{2}} = -\frac{1}{5} = \widehat{h}_S(0),$$

$$\widehat{h}_B(1) = \frac{1}{B_0} \frac{8}{9} \frac{\frac{7}{4}\frac{1}{2}}{\frac{7}{4} - \frac{1}{2}} = \frac{14}{45} \frac{1}{B_0}$$

In conclusion, in order to hedge the derivative the writer must undertake the following strategy: at time $t = 0$, she/he short-sells $\frac{1}{5}$ shares of the stock and buys $\frac{14}{45} B_0^{-1}$ shares of the risk-free asset; note that the initial capital required to open this position is exactly the premium $\widehat{\Pi}_Y(0)$ that the writer received by selling the put option. At time $t = 1$, the writer will do one of the following: if the price of the stock goes up, he/she can just stop hedging (which means that the writer will close the portfolio). If the price of the stock goes down, the writer will strengthen her/his short position on the stock by short-selling even more shares of the stock (in fact, $h_S(2) = -\frac{4}{5} < -\frac{1}{5} = h_S(1)$) and the resulting income will be used to buy more share of the risk-free asset. As shown in the class, if the buyer does not exercise the derivative at the optimal time 1 (when $S(1) = 1/2$), the writer can withdraw the cash $C(1) = 1/36$, and still be able to hedge the derivative a maturity $T = 2$.

Exercise 4.7, Ref. [1]

Consider an American derivative with intrinsic value

$$Y(t) = \min(S(t), (24 - S(t))_+)$$

and expiring at time $T = 3$. The initial price of the underlying stock is $S(0) = 27$, while at future times it follows the binomial model

$$S(t+1) = \begin{cases} 4S(t)/3 & \text{with probability } 1/2 \\ 2S(t)/3 & \text{with probability } 1/2 \end{cases}$$

for $t = 0, 1, 2$. Assume also that the interest rate of the money market is zero. Compute the possible paths of the fair value of the derivative. In which case it is optimal for the buyer to exercise the derivative prior to expiration? What is the amount of cash that the seller can withdraw from the portfolio if the buyer does not exercise the derivative optimally?

Solution

With the given values of the parameters u, d, r, we have

$$q_u = \frac{e^r - e^d}{e^u - e^d} = \frac{1 - \frac{2}{3}}{\frac{4}{3} - \frac{2}{3}} = \frac{1}{2} = q_d.$$

The fair price $\widehat{\Pi}_Y(t)$ of the American derivative satisfies

$$\begin{aligned}
\widehat{\Pi}_Y(t) &= \max(Y(t), e^{-r}(q_u \widehat{\Pi}_Y^u(t+1) + q_d \widehat{\Pi}_Y^d(t+1))) \\
&= \max(Y(t), \frac{1}{2}(\widehat{\Pi}_Y^u(t+1) + \widehat{\Pi}_Y^d(t+1))),
\end{aligned}$$

where $\widehat{\Pi}_Y^u(t)$ (resp. $\widehat{\Pi}_Y^d(t)$) is the price of the derivative at time t assuming that the stock price goes up (resp. down) at time t. The diagram of the stock price is

$$\begin{tikzcd}
 & & & S(3)=64 \\
 & & S(2)=48 \arrow[ur, "u"] \arrow[dr, "d"] & \\
 & S(1)=36 \arrow[ur, "u"] \arrow[dr, "d"] & & S(3)=32 \\
S(0)=27 \arrow[ur, "u"] \arrow[dr, "d"] & & S(2)=24 \arrow[ur, "u"] \arrow[dr, "d"] & \\
 & S(1)=18 \arrow[ur, "u"] \arrow[dr, "d"] & & S(3)=16 \\
 & & S(2)=12 \arrow[ur, "u"] \arrow[dr, "d"] & \\
 & & & S(3)=8
\end{tikzcd}$$

to which there corresponds the following diagram for the intrinsic value:

$$\begin{tikzcd}
 & & & Y(3)=0=\widehat{\Pi}_Y(3) \\
 & & Y(2)=0 \arrow[ur, "u"] \arrow[dr, "d"] & \\
 & Y(1)=0 \arrow[ur, "u"] \arrow[dr, "d"] & & Y(3)=0=\widehat{\Pi}_Y(3) \\
Y(0)=0 \arrow[ur, "u"] \arrow[dr, "d"] & & Y(2)=0 \arrow[ur, "u"] \arrow[dr, "d"] & \\
 & Y(1)=6 \arrow[ur, "u"] \arrow[dr, "d"] & & Y(3)=8=\widehat{\Pi}_Y(3) \\
 & & Y(2)=12 \arrow[ur, "u"] \arrow[dr, "d"] & \\
 & & & Y(3)=8=\widehat{\Pi}_Y(3)
\end{tikzcd}$$

Therefore

$$S(2) = 48 \Rightarrow \widehat{\Pi}_Y(2) = 0, \quad S(2) = 24 \Rightarrow \widehat{\Pi}_Y(2) = 4, \quad S(2) = 12 \Rightarrow \widehat{\Pi}_Y(2) = 12$$

$$S(1) = 36 \Rightarrow \widehat{\Pi}_Y(1) = 2, \quad S(1) = 18 \Rightarrow \widehat{\Pi}_Y(1) = 8,$$

and $\widehat{\Pi}_Y(0) = 5$. We thereby obtained the following diagram for the price of the derivative:

$$\begin{tikzcd}
 & & & \widehat{\Pi}_Y(3) = 0 \\
 & & \widehat{\Pi}_Y(2) = 0 \arrow[ur, "u"] \arrow[dr, "d"] & \\
 & \widehat{\Pi}_Y(1) = 2 \arrow[ur, "u"] \arrow[dr, "d"] & & \widehat{\Pi}_Y(3) = 0 \\
\widehat{\Pi}_Y(0) = 5 \arrow[ur, "u"] \arrow[dr, "d"] & & \widehat{\Pi}_Y(2) = 4 \arrow[ur, "u"] \arrow[dr, "d"] & \\
 & \widehat{\Pi}_Y(1) = 8 \arrow[ur, "u"] \arrow[dr, "d"] & & \widehat{\Pi}_Y(3) = 8 \\
 & & \boxed{\widehat{\Pi}_Y(2) = 12} \arrow[ur, "u"] \arrow[dr, "d"] & \\
 & & & \widehat{\Pi}_Y(3) = 8
\end{tikzcd}$$

This completes the first part of the exercise. The only case in which the price of the derivative equals its intrinsic value prior to expiration is at time $t = 2$ when the price of the stock is $S(2) = 12$ (i.e., the stock price goes down in the first two steps). This is indicated in the previous diagram by putting the price of the derivative in a box. In this case, and only in this case, it is optimal to exercise the derivative prior to expiration. If the buyer does not exercise the derivative at the optimal moment, the writer can withdraw the amount

$$C(2) = \widehat{\Pi}_Y(2) - e^{-r}[q_u\widehat{\Pi}_Y(3)^u + q_d\widehat{\Pi}_Y^d(3)] = 12 - [\frac{1}{2}8 - \frac{1}{2}8] = 4.$$

Note that after withdrawing this cash, the value of the portfolio is 8, which is exactly what the writer needs to hedge the derivative!