Solutions to selected exercises (week 6)

Exercise 6.5, Ref. [1]

Compute the Black-Scholes price of a physically settled binary options.

Solution

The pay-off of a physically settled binary option is $Y = S(T)H(S(T) - K)$, where H is the Heaviside function. Hence, if $S(T) > K$, the buyer of the option receives $S(T)$ (i.e., the buyer receives the stock), while if $S(T) \leq K$ the buyer receives nothing. The pay-off function is

$$g(x) = xH(x-K) = \begin{cases} x & \text{if } x > K \\ 0 & \text{if } x \leq K. \end{cases}$$

The Black-Scholes price is $\Pi_Y(t) = v(t, S(t))$, where

$$v(t,x) = \frac{e^{-r\tau}}{\sqrt{2\pi}} \int_{\mathbb{R}} g\Big(xe^{(r-\frac{\sigma^2}{2})\tau} e^{\sigma\sqrt{\tau}y}\Big) e^{-\frac{y^2}{2}}\, dy$$

see Definition 6.1. Here $\tau = T - t$ is the time left to maturity, r the interest rate of the risk-free asset (i.e., $B(t) = B_0 e^{rt}$, $t \in [0,T]$), and σ is the volatility of the stock. In the integral we use

$$g\Big(xe^{(r-\frac{\sigma^2}{2})\tau} e^{\sigma\sqrt{\tau}y}\Big) = \begin{cases} xe^{(r-\frac{\sigma^2}{2})\tau} e^{\sigma\sqrt{\tau}y} & \text{if } xe^{(r-\frac{\sigma^2}{2})\tau} e^{\sigma\sqrt{\tau}y} > K \\ 0 & \text{if } xe^{(r-\frac{\sigma^2}{2})\tau} e^{\sigma\sqrt{\tau}y} \leq K \end{cases}$$

or equivalently

$$g\Big(xe^{(r-\frac{\sigma^2}{2})\tau} e^{\sigma\sqrt{\tau}}\Big) = \begin{cases} xe^{(r-\frac{\sigma^2}{2})\tau} e^{\sigma\sqrt{\tau}y} & \text{if } y > -d_2 \\ 0 & \text{if } y \leq -d_2 \end{cases}$$

where

$$d_2 = \frac{\log\frac{x}{K} + (r - \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}$$

Hence

$$v(t,x) = \frac{e^{-r\tau}}{\sqrt{2\pi}} \int_{-d_2}^{\infty} x e^{(r-\frac{\sigma^2}{2})\tau} e^{\sigma\sqrt{\tau}y} e^{-\frac{y^2}{2}}\, dy = \frac{x}{\sqrt{2\pi}} \int_{-d_2}^{\infty} e^{-\frac{1}{2}(y-\sigma\sqrt{\tau})^2}\, dy$$
$$= \frac{x}{\sqrt{2\pi}} \int_{-d_2-\sigma\sqrt{\tau}}^{\infty} e^{-\frac{1}{2}z^2}\, dz = \frac{x}{\sqrt{2\pi}} \int_{-\infty}^{d_2+\sigma\sqrt{\tau}} e^{-\frac{1}{2}z^2}\, dz = x\Phi(d_1)$$

where $d_1 = d_2 + \sigma\sqrt{\tau}$.

1 Exercise 6.6, Ref. [1]

Consider a European derivative with maturity T and pay-off Y given by

$$Y = k + S(T)\log S(T),$$

where $k > 0$ is a constant. Find the Black-Scholes price of the derivative at time $t < T$ and the hedging self-financing portfolio. Find the probability that the derivative expires in the money.

Solution

The pay-off function is $g(z) = k + z\log z$. Hence the Black-Scholes price of the derivative is $\Pi_Y(t) = v(t, S(t))$, where

$$v(t,s) = e^{-r\tau} \int_{\mathbb{R}} g\left(s e^{\left(r-\frac{\sigma^2}{2}\right)\tau - \sigma\sqrt{\tau}x} \right) e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}}$$
$$= e^{-r\tau} \int_{\mathbb{R}} \left(k + s e^{(r-\frac{\sigma^2}{2})\tau - \sigma\sqrt{\tau}x} (\log s + (r - \frac{\sigma^2}{2})\tau - \sigma\sqrt{\tau}x) \right) e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}}$$
$$= k e^{-r\tau} + s\log s \int_{\mathbb{R}} e^{-\frac{1}{2}(x+\sigma\sqrt{\tau})^2} \frac{dx}{\sqrt{2\pi}}$$
$$+ s(r - \frac{\sigma^2}{2})\tau \int_{\mathbb{R}} e^{-\frac{1}{2}(x+\sigma\sqrt{\tau})^2} \frac{dx}{\sqrt{2\pi}} - s\sigma\sqrt{\tau} \int_{\mathbb{R}} x e^{-\frac{1}{2}(x+\sigma\sqrt{\tau})^2} \frac{dx}{\sqrt{2\pi}}$$

Using that

$$\int_{\mathbb{R}} e^{-\frac{1}{2}(x+\sigma\sqrt{\tau})^2} \frac{dx}{\sqrt{2\pi}} = 1, \quad \int_{\mathbb{R}} x e^{-\frac{1}{2}(x+\sigma\sqrt{\tau})^2} \frac{dx}{\sqrt{2\pi}} = -\sigma\sqrt{\tau},$$

we obtain

$$v(t,s) = k e^{-r\tau} + s\log s + s(r + \frac{\sigma^2}{2})\tau.$$

Hence

$$\Pi_Y(t) = k e^{-r\tau} + S(t)\log S(t) + S(t)(r + \frac{\sigma^2}{2})\tau.$$

This completes the first part of the exercise. The number of shares of the stock in the hedging portfolio is given by

$$h_S(t) = \Delta(t, S(t)),$$

where $\Delta(t, s) = \frac{\partial v}{\partial s} = \log s + 1 + (r + \frac{\sigma^2}{2})\tau$. Hence

$$h_S(t) = 1 + (r + \frac{\sigma^2}{2})\tau + \log S(t).$$

The number of shares of the risk-free asset is obtained by using that

$$\Pi_Y(t) = h_S(t)S(t) + B(t)h_B(t),$$

hence

$$\begin{aligned}
h_B(t) &= \frac{1}{B(t)}(\Pi_Y(t) - h_S(t)S(t)) \\
&= e^{-rt}(ke^{-r\tau} + S(t)\log S(t) + S(t)(r + \frac{\sigma^2}{2})\tau - S(t) - S(t)(r + \frac{\sigma^2}{2})\tau - S(t)\log S(t)) \\
&= ke^{-rT} - S(t)e^{-rt}.
\end{aligned}$$

This completes the second part of the exercise. To compute the probability that $Y > 0$, we first observe that the pay-off function $g(z)$ has a minimum at $z = e^{-1}$ and we have $g(e^{-1}) = k - e^{-1}$. Hence if $k \geq e^{-1}$, the derivative has probability 1 to expire in the money. If $k < e^{-1}$, there exist $a < b$ such that

$$g(z) > 0 \quad \text{if and only if} \quad 0 < z < a \text{ or } z > b.$$

Hence for $k < e^{-1}$ we have

$$\mathbb{P}(Y > 0) = \mathbb{P}(S(T) < a) + \mathbb{P}(S(T) > b).$$

Since $S(T) = S(0)e^{\alpha T - \sigma\sqrt{T}G}$, with $G \in N(0, 1)$, then

$$S(T) < a \Leftrightarrow G > \frac{\log \frac{S(0)}{a} + \alpha T}{\sigma\sqrt{T}} := A, \quad S(t) > b \Leftrightarrow G < \frac{\log \frac{S(0)}{b} + \alpha T}{\sigma\sqrt{T}} := B.$$

Thus

$$\begin{aligned}
\mathbb{P}(Y > 0) = \mathbb{P}(G > A) + \mathbb{P}(G < B) &= \int_A^{+\infty} e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} + \int_{-\infty}^{B} e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} \\
&= 1 - \Phi(A) + \Phi(B).
\end{aligned}$$

This completes the solution of the third part of the exercise.