Hints for recommended exercises in Ref. [1] (week 7)

Exercise 6.2

Proceed as in the proof of Theorem 6.2, where the Black-Scholes price is computed for call options

Exercise 6.3

The put-call parity is

$$C(t, S(t)) - P(t, S(t)) = S(t) - Ke^{-r(T-t)}$$

where C is the price of the call, P the price of the put, $S(t)$ the price of the stock, K the strike of the call and put option, T the maturity of the options and r the interest rate of the risk-free asset. Hence we have for instance

$$\partial_x P = \partial_x C - 1 = \Phi(d_1) - 1 = \int_{-\infty}^{d_1} \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}\, dy - \int_{-\infty}^{\infty} \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}\, dy = -\int_{d_1}^{\infty} \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}\, dy = -\Phi(-d_1)$$

Exercise 6.4

Let $C(t, x)$ be the Black-Scholes price function of European calls. Prove that

$$\lim_{\sigma\to 0^+} C(t, x) = (x - Ke^{-r\tau})_+, \quad \lim_{\sigma\to\infty} C(t, x) = x.$$

Compute also the following limits:

$$\lim_{K\to 0^+} C(t, x), \qquad \lim_{K\to +\infty} C(t, x), \qquad \lim_{T\to +\infty} C(t, x), \qquad \lim_{x\to 0^+} C(t, x).$$

Repeat all the above for put options.

Solution

Recall that

$$C(t, x) = x\Phi(d_1) - Ke^{-r\tau}\Phi(d_2), \tag{1}$$

where

$$d_2 = \frac{\log\left(\frac{x}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}, \quad d_1 = d_2 + \sigma\sqrt{\tau}, \tag{2}$$

and where $\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{1}{2}y^2}\,dy$ is the standard normal distribution. As $\sigma \to 0^+$ we have $d_1 \to d_2$ and

$$d_2 \sim \frac{1}{\sqrt{\tau}}(\log\frac{x}{K} + r\tau)\sigma^{-1}.$$

Hence

$$\begin{aligned}
&d_2 \to +\infty, \quad \text{if } x > Ke^{-r\tau},\\
&d_2 \to -\infty, \quad \text{if } x < Ke^{-r\tau},\\
&d_2 \to 0, \quad \text{if } x = Ke^{-r\tau},
\end{aligned}$$

Thus

$$\begin{aligned}
&\lim_{\sigma\to 0^+}\Phi(d_1) = \lim_{\sigma\to 0^+}\Phi(d_2) = 1, \quad \text{if } x > Ke^{-r\tau},\\
&\lim_{\sigma\to 0^+}\Phi(d_1) = \lim_{\sigma\to 0^+}\Phi(d_2) = 0, \quad \text{if } x < Ke^{-r\tau},\\
&\lim_{\sigma\to 0^+}\Phi(d_1) = \lim_{\sigma\to 0^+}\Phi(d_2) = \Phi(0), \quad \text{if } x = Ke^{-r\tau}.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\lim_{\sigma\to 0^+} C(t,x) = x - Ke^{-r\tau} \quad \text{if } x > Ke^{-r\tau},\\
&\lim_{\sigma\to 0^+} C(t,x) = 0, \quad \text{if } x \le Ke^{-r\tau},
\end{aligned}$$

i.e., $\lim_{\sigma\to 0^+} C(t,x) = (x - Ke^{-r\tau})_+$. For $\sigma \to +\infty$ we have $d_2 \to -\infty$ and $d_1 \to +\infty$, hence $\Phi(d_1) \to 1$ and $\Phi(d_2) \to 0$. Thus $C(t,x) \to x$ as $\sigma \to +\infty$. As $K \to 0^+$, both d_1 and d_2 diverge to $+\infty$, hence

$$\lim_{K\to 0^+} C(t,x) = x.$$

For $K \to +\infty$, d_1, d_2 diverge to $-\infty$. Hence the first term in $C(t,x)$ converges to zero. As the first term in $C(t,x)$ always dominates the second term (since $C(t,x) > 0$), then the second term also goes to zero and thus

$$\lim_{K\to +\infty} C(t,x) = 0.$$

For $T \to +\infty$ we obtain

$$\lim_{T\to +\infty} C(t,x) = x.$$

Finally, for $x \to 0^+$, both d_1, d_2 diverge to $-\infty$ and thus

$$\lim_{x\to 0^+} C(t,x) = 0.$$

To compute the limits for put options we use the put-call parity:

$$C(t,x) - P(t,x) = x - Ke^{-r\tau},$$

by which it follows that

$$\begin{aligned}
&\lim_{\sigma\to 0^+} P(t,x) = (Ke^{-r\tau} - x)_+, \quad \lim_{\sigma\to+\infty} P(t,x) = Ke^{-r\tau} \\
&\lim_{K\to 0^+} P(t,x) = 0, \quad \lim_{K\to+\infty} P(t,x) = +\infty, \\
&\lim_{T\to+\infty} P(t,x) = 0, \quad \lim_{x\to 0^+} P(t,x) = Ke^{-r\tau}.
\end{aligned}$$

Ref. [2], Sec. 5.3, Ex. 5

This exercise will be solved in the class on Tuesday 24 May (this extra lecture will be held in Euler at 13.15 as a preparation for the exam.)