Exam for the course "Options and Mathematics" (CTH[*MVE095*], GU[*MMA700*]) 2016/17

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REMARKS: (1) No aids permitted (2) Minor errors in the calculations will be forgiven, but remember that fractions look nicer when you simplify them!

1. (i) Define and explain the concept of self-financing portfolio process invested in a binomial market (max. 1 point)

   (ii) Show that, in a binomial market, the portfolio process given by

$$h_S(t) = \frac{1}{S(t-1)}\frac{\Pi_Y^u(t) - \Pi_Y^d(t)}{e^u - e^d}, \quad h_B(t) = \frac{e^{-r}}{B(t-1)}\frac{e^u\Pi_Y^d(t) - e^d\Pi_Y^u(t)}{e^u - e^d},$$

   for $t \in \{1, \dots, N\}$, is a self-financing, predictable, hedging portfolio for the standard European derivative with pay off Y and maturity $T = N$ (max. 4 points).

   Solution. Definition 2.3 and Theorem 3.3 in the lecture notes. Self-financing portfolio means that no cash can ever be removed or added to the portfolio.

2. Consider a 3-period binomial model with the following parameters:

$$e^u = \frac{5}{4}, \quad e^d = \frac{1}{2}, \quad e^r = 1, \quad p \in (0,1).$$

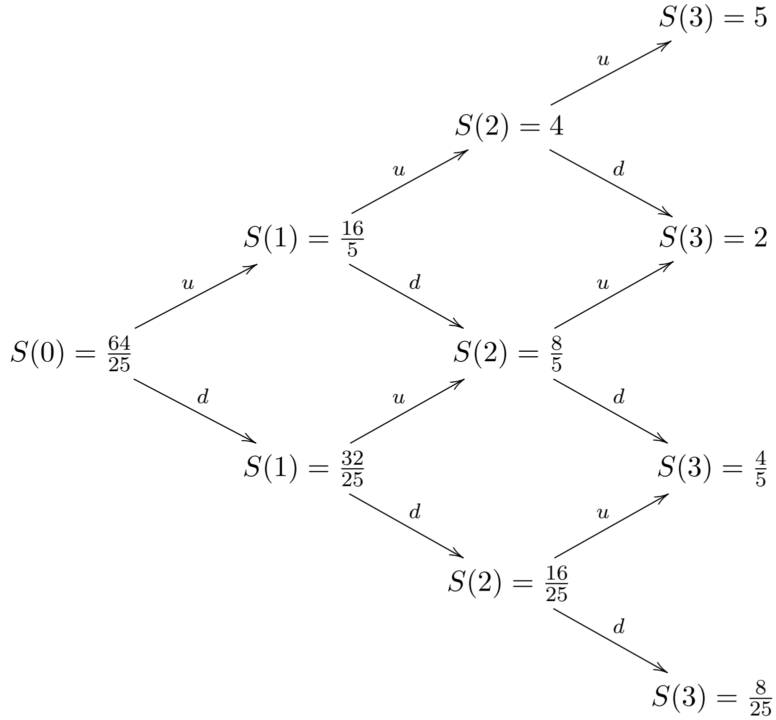
   Let $S(0) = \frac{64}{25}$ be the initial price of the stock. Consider an American style derivative on the stock with maturity $T = 3$ and intrinsic value

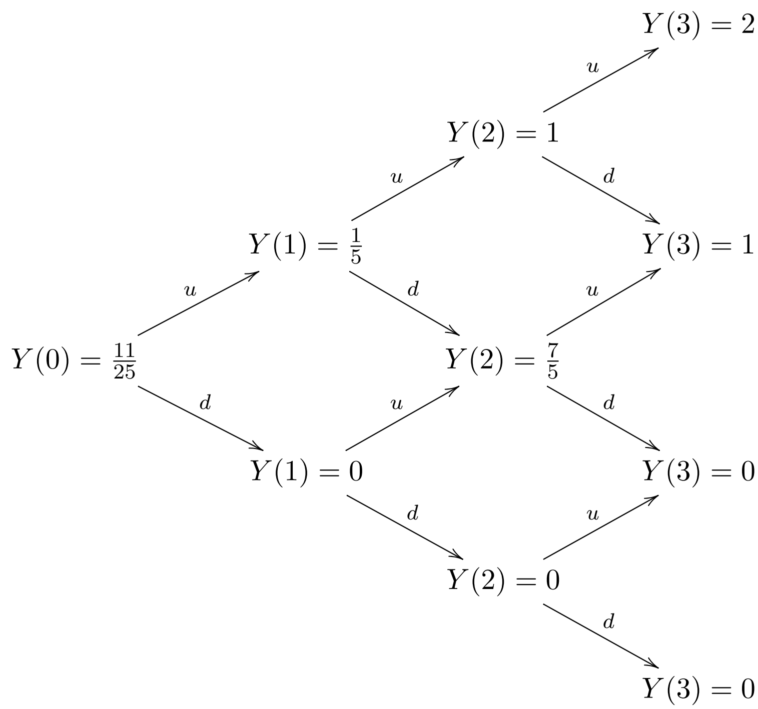
$$Y(t) = \big|3 - S(t)\big|\, H\big(S(t) - 7/5\big),$$

   where $H(x)$ is the Heaviside function and $|x|$ is the absolute value of x (recall that $H(x) = 1$ if $x \geq 0$, $H(x) = 0$ if $x < 0$).

   Compute the binomial price of the derivative at each time $t \in \{0, 1, 2, 3\}$ (max. 1 point) and the initial position on the stock in the hedging portfolio (max. 1 point). Compute the cash that the seller can withdraw from the portfolio if the buyer does not exercise the derivative at optimal times (max. 1 point). Compute the probability that the derivative is in the money at time t (max. 1 point) and the probability that the return for the buyer is positive at time t (max. 1 point), where $t \in \{0, 1, 2, 3\}$.

Solution. The binomial trees for the stock price $S(t)$ and the intrinsic value $Y(t)$ are as follows.

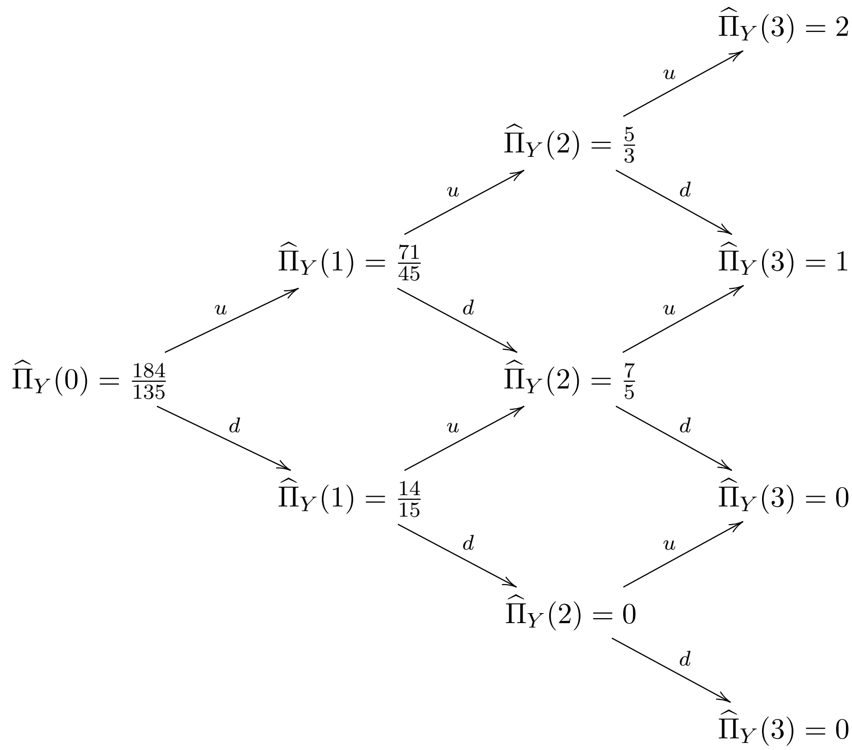




The binomial price of the American derivative is defined as

$$\widehat{\Pi}_Y(3) = Y(3), \quad \widehat{\Pi}_Y(t) = \max[Y(t), e^{-r}(q_u \widehat{\Pi}_Y^u(t+1) + q_u \widehat{\Pi}_Y^u(t+1))], \ t = 0, 1, 2,$$

where $q_u = 2/3$ and $q_d = 1/3$. Applying the above formula one finds the following binomial tree for $\widehat{\Pi}_Y(t)$.



This concludes the first part of the exercise (1 point). The initial position on the stock in the hedging portfolio is

$$h_S(0) = h_S(1) = \frac{1}{S(0)} \frac{\widehat{\Pi}_Y^u(1) - \widehat{\Pi}_Y^d(1)}{e^u - e^d} = \frac{1}{\frac{64}{25}} \frac{\frac{71}{45} - \frac{14}{15}}{\frac{3}{4}} = \frac{145}{432},$$

which answers the second question (1 point). As to the cash flow, observe that the only optimal exercise time is $t = 2$ when $S(2) = 8/5$, as in this case, and only in this case, $\widehat{\Pi}_Y(t)$ and $Y(t)$ are equal. If the buyer does not exercise the derivative at this instance, the seller can withdraw the amount

$$C(2) = \widehat{\Pi}_Y(2) - e^{-r}(q_u \widehat{\Pi}_Y^u(t+1) + q_u \widehat{\Pi}_Y^u(t+1)) = \frac{7}{5} - \frac{2}{3} = \frac{11}{15}.$$

This answers the third question (1 point). The probability that the derivative is in the money at time t is $\mathbb{P}(Y(t) > 0)$. We have

$$\mathbb{P}(Y(0) > 0) = 1, \quad \mathbb{P}(Y(1) > 0) = p,$$
$$\mathbb{P}(Y(2) > 0) = p^2 + 2p(1-p) = p(2-p), \quad \mathbb{P}(Y(3) > 0) = p^2 + 3p^2(1-p) = p^2(3-2p).$$

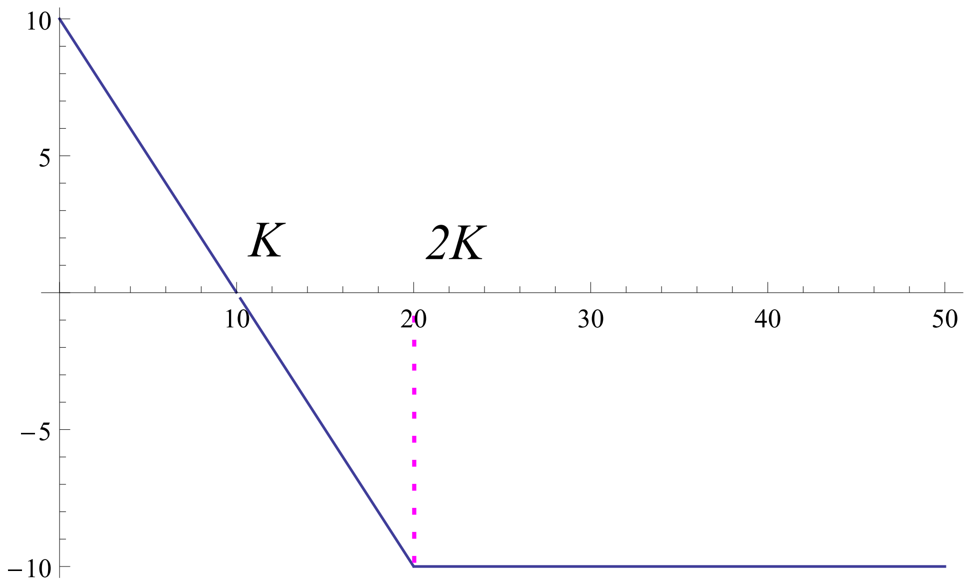
This answers the fourth question (1 point). Finally, the return for the buyer at time t is positive if and only if $Y(t) > \Pi_Y(0)$ (the question is relevant because the buyer can exercise at any time). This happens only at time t when $S(2) = 8/5$ (the optimal exercise time) and at maturity $t = 3$ when $S(3) = 5$. Hence we have

$$\mathbb{P}(R(0) > 0) = 0, \quad \mathbb{P}(R(1) > 0) = 0,$$
$$\mathbb{P}(R(2) > 0) = 2p(1-p) \quad \mathbb{P}(R(3) > 0) = p^3.$$

This completes the exercise (1 point).

3. Let $K > 0$. A European style derivative on a stock with maturity $T > 0$ gives to its owner the right to choose between selling the stock for the price K at time T or paying the amount K at time T. Draw the pay-off function of the derivative (max. 1 point). Compute the Black-Scholes price of the derivative (max. 1 point). Show that there exists a value $K_* > 0$ of K such that the Black-Scholes price of the derivative is zero. What is the financial interpretation of K_*? (max. 3 points).

Solution: The graph of the pay-off looks like in the following picture (the numbers on the axes are irrelevant).



This answers the first question (1 point). By this picture one can see that

$$g(x) = (K - x)_+ - (x - K)_+ + (x - 2K)_+ = g_1(x) + g_2(x) + g_3(x).$$

As the Black-Scholes price is linear in the pay-off function, the Black-Scholes price of the derivative is the sum of the Black-Scholes price of the derivatives with pay-off functions g_1, g_2, g_3, hence

$$\Pi_Y(t) = P(t, S(t), K, T) - C(t, S(t), K, T) + C(t, S(t), 2K, T)$$
$$= Ke^{-r\tau} - S(t) + C(t, S(t), 2K, T),$$

where for the second equality we used the put-call parity. Here $C(t, S(t), K, T)$ denotes the Black-Scholes price of the European call with strike K and maturity T. Hence

$$C(t, S(t), 2K, T) = S(t)\Phi(\widetilde{d}_1) - 2Ke^{-r\tau}\Phi(\widetilde{d}_2),$$

where

$$\widetilde{d}_1 = \frac{\log \frac{S(t)}{2K} + (r + \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}, \quad \widetilde{d}_2 = \frac{\log \frac{S(t)}{2K} + (r - \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}.$$

This completes the second part of the exercise (1 point). For the last part, denote

$$f(K) = Ke^{-r\tau} - S(t) + C(t, S(t), 2K, T)$$

the price of the derivative as a function of K. We want to prove that there exists $K_* > 0$ such that $f(K_*) = 0$. First we observe that $f(K) > 0$ for $K > S(t)e^{r\tau}$. Hence, if we prove that $f(K)$ can also take negative values, then, since f is continuous, there must exist K_* such that $f(K_*) = 0$. To this purpose define K_0 such that $\widetilde{d}_2 = 0$, that is

$$K_0 = \frac{1}{2}S(t)e^{(r-\frac{\sigma^2}{2})\tau}.$$

For this strike price we have $\Phi(\widetilde{d}_2) = 1/2$ and so the function f evaluated at K_0 is

$$f(K_0) = (\Phi(\widetilde{d}_1) - 1)S(t).$$

As $\Phi(\widetilde{d}_1) < 1$, we find $f(K_0) < 0$. Hence there exists $K_0 < K_* < S(t)e^{r\tau}$ such that $f(K_*) = 0$. The value K_* is the fair value of the strike price for the derivative. In fact, as both the buyer and the seller of this derivative can loose money at maturity, the fair price of the derivative should be zero. This concludes the solution of the exercise (3 points).