Flervariabelanalys AT
MVE115
Lösningar 13/3 -10

1. $f(r,s,t)$, $r = x-y$
$s = y-z$
$t = z-x$

$$\frac{\partial u}{\partial x} = \frac{\partial f}{\partial r}\underbrace{\frac{\partial r}{\partial x}}_{=1} + \frac{\partial f}{\partial s}\underbrace{\frac{\partial s}{\partial x}}_{=0} + \frac{\partial f}{\partial t}\underbrace{\frac{\partial t}{\partial x}}_{=-1} = \frac{\partial f}{\partial r} - \frac{\partial f}{\partial t}$$

$$\frac{\partial u}{\partial y} = \frac{\partial f}{\partial r}\cdot(-1) + \frac{\partial f}{\partial s}\cdot 1 + \frac{\partial f}{\partial t}\cdot 0 = -\frac{\partial f}{\partial r} + \frac{\partial f}{\partial s}$$

$$\frac{\partial u}{\partial z} = \frac{\partial f}{\partial r}\cdot 0 + \frac{\partial f}{\partial s}\cdot(-1) + \frac{\partial f}{\partial t}\cdot 1 = -\frac{\partial f}{\partial s} + \frac{\partial f}{\partial t}$$

$$\Rightarrow \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$$

---

2. (a) $v_0 = \frac{v}{|v|} = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)$

$$\nabla f\Big|_{(1,1,0)} = (2x-2, 4y, 6z+12)\Big|_{(1,1,0)} = (0,4,12)$$

$$D_{v_0} f(1,1,0) = 0\cdot\frac{1}{\sqrt{2}} + 4\cdot\left(-\frac{1}{\sqrt{2}}\right) + 12\cdot 0 = -2\sqrt{2}$$

(b) $f = 15$ på ytan $\Rightarrow$ det är en nivåyta till f

$\nabla f(P)$ normalvektor till ytan i $P \in$ ytan

tangentplanet: $\nabla f(P)\cdot(x-0, y-0, z-1)$

$P(0,0,1)$

$\Rightarrow$ ekvationen för tangentplanet (2) är

$$-2x + 0\cdot y + 18(z-1), \text{ d.v.s.}$$
$$x - 9z = -9$$

(3.) (a) Det är $f(x,y,z) = \frac{xy}{z}$

$$f(2,1,2) + \frac{\partial f}{\partial x}(2,1,2)(x-2) +$$
$$+ \frac{\partial f}{\partial y}(2,1,2)(y-1) + \frac{\partial f}{\partial z}(2,1,2)(z-2) =$$
$$= 1 + \frac{1}{2}\Delta x + \Delta y - \frac{1}{2}\Delta z,$$ där
$$\Delta x = x-2, \quad \Delta y = y-1, \quad \Delta z = z-2$$

(b) Om $|AB| = x$, $|BC| = z$,
$|\text{höjden mot } AB| = y$, så är
$h = |\text{höjden mot } BC| = f(x,y,z) = \frac{xy}{z}$

x har uppmätts till 2
y —— " —— 1
z —— " —— 2

ger $f(2,1,2) = 1$

$$\Rightarrow |h-1| \lesssim \frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{100} + \frac{1}{100} + \frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{100} = \frac{3}{100}$$
$$(= 3\%)$$

(4.) γ är randen till en rektangel D

γ styckvis C^1 (C^1 utom i hörnen)

$$F = (\underbrace{e^{x^2} - x\cos y}_{P}, \underbrace{\sin y^2 + x\sin y}_{Q}) \in C^\infty$$

$\Rightarrow$ Greens sats kan användas

γ orienterad medurs

$$\Rightarrow \quad -\int_\gamma F\cdot dr = \iint_D \left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dx\,dy =$$

$$= \iint_D (\sin y + x\sin y)\,dx\,dy =$$

$$= \int_{-1}^{1}\int_0^{\pi}(x+1)\sin y\,dx\,dy =$$

$$= \int_{-1}^{1}\left(\int_0^{\pi} (x+1)\sin y\,dy\right)dx =$$

$$= \int_{-1}^{1} (x+1)\,dx \cdot \int_0^{\pi} \sin y\,dy =$$

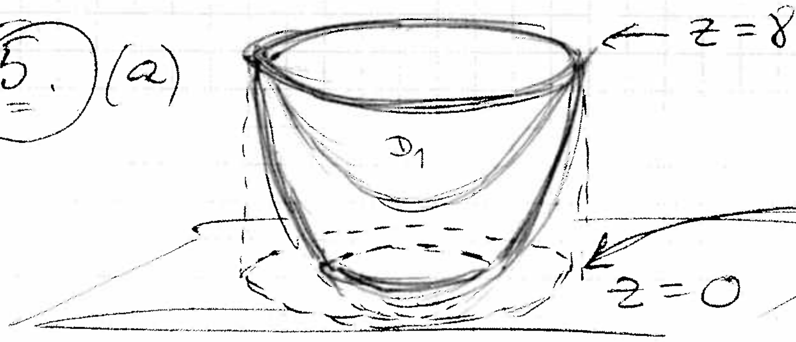
$$= \left[\frac{1}{2}(x+1)^2\right]_{-1}^{1} \cdot \left[-\cos y\right]_0^{\pi} =$$

$$= 2\cdot 2 = 4$$

$$\Rightarrow \quad \int_\gamma F\cdot dr = -4$$

---

5. (a)



$$x^2+y^2-1 = 2+\frac{2}{3}(x^2+y^2)$$

$$\Leftrightarrow \quad x^2+y^2 = 3^2$$

(ger $z = 8$)

(b) $V_{vatten} = \iiint_{D_1} 1\cdot dx\,dy\,dz =$

$$= \iint_{x^2+y^2\le 3^2} \left(8-\left(2+\frac{2}{3}(x^2+y^2)\right)\right)dx\,dy =$$

$$= \int_0^3 \int_0^{2\pi} \left(6 - \frac{2}{3}r^2\right) r \, d\varphi \, dr =$$

$$= 2\pi \left[3r^2 - \frac{1}{6}r^4\right]_0^3 =$$

$$= 2\pi \left(27 - \frac{27 \cdot \cancel{3}}{2 \cdot \cancel{3}}\right) = 27\pi$$

(b) $z = 0$ skär $z = x^2 + y^2 - 1$ i enhetscirkeln

Den del av paraboloiden $z = x^2 + y^2 - 1$ som projiceras på enhetsskivan ligger under xy-planet ($z = 0$)

$$\Rightarrow V_{glas} = \iint_{1^2 \leq x^2+y^2 \leq 3^2} \left(2 + \frac{2}{3}(x^2+y^2) - (x^2+y^2-1)\right) dx\,dy$$

$$+ \iint_{x^2+y^2 \leq 1} \left(2 + \frac{2}{3}(x^2+y^2) - 0\right) dx\,dy =$$

$$= \int_1^3 \int_0^{2\pi} \left(3 - \frac{1}{3}r^2\right) r \, d\varphi \, dr + \int_0^1 \int_0^{2\pi} \left(2 + \frac{2}{3}r^2\right) r \, d\varphi \, dr$$

$$= 2\pi \left(\left[\frac{3}{2}r^2 - \frac{1}{12}r^4\right]_1^3 + \left[r^2 + \frac{1}{6}r^4\right]_0^1 \right) =$$

$$= 2\pi \left(\frac{27}{2} - \frac{\cancel{81}^{27}}{\cancel{12}_4} - \frac{3}{2} + \frac{1}{12} + 1 + \frac{1}{6} - 0 \right) =$$

$$= 2\pi \, \frac{81 - 18 + 1 + 12 + 2}{12} = 2\pi \cdot \frac{\cancel{78}^{13}}{\cancel{12}_2} =$$

$$= 13\pi$$

5

6. (a) max $(x+y+z^2)$
när $x^2+y^2+z^2=1$

(1) Lagranges multiplikatormetod

$$\Phi(x,y,z) = x+y+z^2-\lambda(x^2+y^2+z^2-1)$$

$$\begin{cases} \nabla\Phi = 0 \\ x^2+y^2+z^2=1 \end{cases}$$

$\exists$ max, ty sfären kompakt
man jämför f:s värden i
lösningarna till ekvationssystemet;
det största ger max f

(2) på ~~sfären~~ : $z^2 = 1-x^2-y^2$
$x^2+y^2 \le 1$

$$\max(\underbrace{x+y+1-x^2-y^2}_{\varphi(x,y)}) \quad \text{på } x^2+y^2\le 1$$

Stationära pkter: $\frac{\partial\varphi}{\partial x}=0$, $\frac{\partial\varphi}{\partial y}=0$

på $x^2+y^2=1$: $\varphi = x+y+0$

$x=\cos t$, $y=\sin t$

$\psi(t) = \cos t + \sin t$, $t\in[0,2\pi]$

man jämför φ:s värden i de stationära pkter som ligger i enhetsskivan med ψ:s max i $[0,2\pi]$

(b) $\varphi = x+y+1-x^2-y^2 = \frac{3}{2}-(x-\frac{1}{2})^2-(y-\frac{1}{2})^2$

$\Rightarrow \varphi \le \varphi(\frac{1}{2},\frac{1}{2}) = \frac{3}{2}$, $\frac{1}{4}+\frac{1}{4}=\frac{1}{2}<1$

$\Rightarrow f_{max}$ på sfären $= f(\frac{1}{2},\frac{1}{2},\frac{1}{\sqrt{2}}) = \frac{3}{2}$