MVE115 Flervariabelanalys AT

Lösningar 12/3 - 2011

1. $\dfrac{x^2+y^2}{2x-y} = C$ (1) $C=0 \Leftrightarrow x^2+y^2=0$ origo

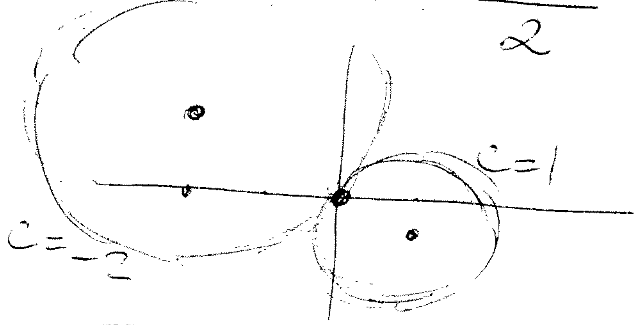
(2) $C \neq 0$ $x^2+y^2 = 2Cx - Cy$

$x^2 - 2Cx + y^2 + Cy = 0$

$(x-C)^2 + \left(y+\frac{C}{2}\right)^2 - C^2 - \frac{C^2}{4} = 0$

cirkel med medelpkt $\left(C, -\frac{C}{2}\right)$

och radie $\dfrac{|C|\sqrt{5}}{2}$ (alla går genom origo)



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2. Polär substitution: $\left|\dfrac{r^5\cos^5\varphi}{r^4(\cos^4\varphi+\sin^4\varphi)}\right| =$

$= \left|\dfrac{r^{\cancel{5}}\cos^5\varphi}{\cancel{r^4}\left(\underbrace{(\cos^2\varphi+\sin^2\varphi)^2}_{=1}-2\cos^2\varphi\sin^2\varphi\right)}\right| =$

$= r\left|\dfrac{\cos\varphi}{1-\frac{1}{2}\sin^2 2\varphi}\right| \leq r\dfrac{1}{1/2} = 2r \to 0$ $\quad \left(1-\frac{1}{2}\sin^2 2\varphi \geq \frac{1}{2}\right)$

$\Rightarrow$ gränsvärdet finns och

$\lim\limits_{(x,y)\to(0,0)} \dfrac{x^5}{x^4+y^4} = 0.$

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3. $\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases}$ Derivera (implicit) m.a.p. x och y

m.a.p. $\underline{\underline{x}}$: $\left|\begin{array}{l} 1 = \dfrac{\partial r}{\partial x}\cos\varphi - r\sin\varphi\dfrac{\partial\varphi}{\partial x} \\ 0 = \dfrac{\partial r}{\partial x}\sin\varphi + r\cos\varphi\dfrac{\partial\varphi}{\partial x} \end{array}\right.$

LES, lös ut $\dfrac{\partial r}{\partial x}$, $\dfrac{\partial\varphi}{\partial x}$

$$\frac{\partial r}{\partial x} = \cos\varphi \quad , \quad \frac{\partial\varphi}{\partial x} = -\frac{1}{r}\sin\varphi$$

m.a.p. $\underline{\underline{y}}$: $\left|\begin{array}{l} 0 = \dfrac{\partial r}{\partial y}\cos\varphi - r\sin\varphi\dfrac{\partial\varphi}{\partial y} \\ 1 = \dfrac{\partial r}{\partial y}\sin\varphi + r\cos\varphi\dfrac{\partial\varphi}{\partial y} \end{array}\right.$

LES, lös ut $\dfrac{\partial r}{\partial y}$, $\dfrac{\partial\varphi}{\partial y}$

$$\frac{\partial r}{\partial y} = \sin\varphi \quad , \quad \frac{\partial\varphi}{\partial y} = \frac{1}{r}\cos\varphi$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial\varphi}\frac{\partial\varphi}{\partial x} =$$

$$= \cos\varphi\frac{\partial u}{\partial r} - \frac{1}{r}\sin\varphi\frac{\partial u}{\partial\varphi}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial\varphi}\frac{\partial\varphi}{\partial y} =$$

$$= \sin\varphi\frac{\partial u}{\partial r} + \frac{1}{r}\cos\varphi\frac{\partial u}{\partial\varphi}$$

$$x\frac{\partial u}{\partial y} - y\frac{\partial u}{\partial x} = \frac{1}{\cancel{r}}\cancel{r}\cos^2\varphi\frac{\partial u}{\partial\varphi} + \cancel{r\cos\varphi\sin\varphi\frac{\partial u}{\partial r}} +$$

$$+ \cancel{r}\frac{1}{\cancel{r}}\sin^2\varphi\frac{\partial u}{\partial\varphi} - \cancel{r\sin\varphi\cos\varphi\frac{\partial u}{\partial r}} = \frac{\partial u}{\partial\varphi} = 0$$

$\Rightarrow$ $u = u(r)$ (konstant m.a.p. $\underline{\varphi}$)

$\Rightarrow$ $u = \Phi(\sqrt{x^2+y^2})$, $\Phi \in C^1$

(eller $\Psi(x^2+y^2)$, $\Psi \in C^1$)

(4.) $x>0,\ y>0$ /3

$$\frac{\partial f}{\partial x} = 3x^2y^2(6-x-y) - x^3y^2 = 0$$

$$\frac{\partial f}{\partial y} = 2x^3y(6-x-y) - x^3y^2 = 0$$

$\left.\begin{matrix} x>0 \\ y>0 \end{matrix}\right\} \Rightarrow x,y \neq 0$ $\quad \left| \begin{matrix} 18-3x-3y-x=0 \\ 12-2x-2y-y=0 \end{matrix}\right.$

LES $\quad \left| \begin{matrix} 4x+3y=18 \\ 2x+3y=12 \end{matrix}\right.$ lösning $\quad \begin{matrix} x=3 \\ y=2 \end{matrix}$

$\Rightarrow$ f har en stationär pkt i den öppna första kvadranten, $(3,2)$

$$\frac{\partial f}{\partial x} = 18x^2y^2 - 4x^3y^2 - 3x^2y^3$$

$$\frac{\partial f}{\partial y} = 12x^3y - 2x^4y - 3x^3y^2$$

$$\left.\frac{\partial^2 f}{\partial x^2}\right|_{(3,2)} = 36xy^2 - 12x^2y^2 - 6xy^3\Big|_{(3,2)} =$$

$$= 36\cdot12 - 12\cdot3\cdot12 - 6\cdot12\cdot2 =$$

$$= 144(\cancel{3} - \cancel{3} - 1) = -144$$

$$\left.\frac{\partial^2 f}{\partial x\partial y}\right|_{(3,2)} = 36x^2y - 8x^3y - 9x^2y^2\Big|_{(3,2)} =$$

$$= 36\cdot18 - 8\cdot3\cdot18 - 9\cdot18\cdot2 =$$

$$= 18(36-24-18) = -108$$

$$\left.\frac{\partial^2 f}{\partial y^2}\right|_{(3,2)} = \frac{36}{3}x^3 - 2x^4 - 6x^3y\Big|_{(3,2)} =$$

$$= \cancel{12\cdot27} - 2\cdot81 - \cancel{12\cdot27} = -162$$

$$Q(h,k) = -\left(144h^2 + 216hk + 162k^2\right) =$$ 4

$$= -9\left(16h^2 + 24hk + 18k^2\right) =$$

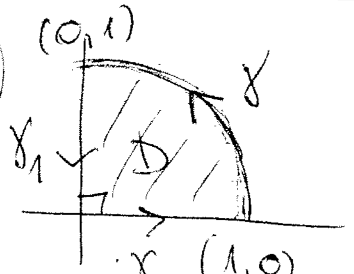
$$= -\underset{\substack{\vee \\ 0}}{\blacksquare}\left((4h+3k)^2 + 9k^2\right) < 0$$

$$\forall (h,k) \neq (0,0)$$

$$\left(\begin{cases} 4h+3k=0 \\ 3k=0 \end{cases} \Rightarrow h=k=0 \right)$$

$\Rightarrow f$ har lok. max i $(3,2)$

5.



Vi sluter till kvartscirkeln γ med sträckorna γ_1 (från $(0,1)$ till $(0,0)$) och γ_2 (från $(0,0)$ till $(1,0)$) för att kunna använda Greens sats

$\gamma_1 : x=0,\ 0 \leq y \leq 1$

$\gamma_2 : y=0,\ 0 \leq x \leq 1$

$\Gamma = \gamma \cup \gamma_1 \cup \gamma_2$ (ett varv moturs)

D = kvartscirkelskivan med rand $\partial D = \Gamma$

$F \in C^\infty(\mathbb{R}^2) (\subset C^1(\mathbb{R}^2))$

Γ styckvis C^1

$\Rightarrow$ vi kan använda Greens sats

$$\int_\Gamma \mathbf{F} \cdot d\mathbf{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \iint_D 2x\,dx\,dy =$$

polära koordinater

$$= \int_0^{\frac{\pi}{2}} \left(\int_0^1 2r\cos\varphi \cdot r\,dr \right) d\varphi = \int_0^{\frac{\pi}{2}} \cos\varphi\,d\varphi \cdot \left[\frac{2}{3}r^3\right]_0^1$$

$$= \left[\sin\varphi\right]_0^{\frac{\pi}{2}} \cdot \frac{2}{3} = \frac{2}{3}$$

$$\int_\gamma = \int_\Gamma - \int_{\gamma_1} - \int_{\gamma_2} \quad ; \quad \int_{\gamma_1} \mathbf{F} \cdot d\mathbf{r} = \int_1^0 e^y\,dy = 1 - e$$

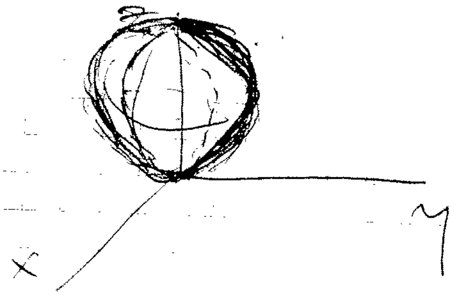
$$\int_{\gamma_2} = \int_0^1 x^3\,dx = \frac{1}{4} \quad \Rightarrow \int_\gamma \mathbf{F} \cdot d\mathbf{r} = \frac{2}{3} - 1 + e - \frac{1}{4} = e - \frac{7}{12}$$

⑥ $z \geq 0$; $z = 0$: endast origo /5

symmetri m.a.p. xz- och yz-planen
($\pm x$ o $\pm y$ ger samma)
funktionen beror av x, y endast
via $r = \sqrt{x^2+y^2}$ $\Rightarrow$ rotationsyta
(rotation kring z-axeln)

Sfäriska koordinater:

$r^4 = r\cos\theta$ $\qquad r = (\cos\theta)^{1/3}$



$\theta = 0 \Rightarrow r = 1$

$\theta = \frac{\pi}{2} \Rightarrow r = 0$

$0 \leq \theta \leq \frac{\pi}{2}$: r avtagande
(avtar långsammare i början)

$$V = \iiint_K dx\,dy\,dz = \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^{(\cos\theta)^{1/3}} r^2 \sin\theta \, dr\, d\theta\, d\varphi$$

$$= 2\pi \int_0^{\frac{\pi}{2}} \sin\theta \cdot \frac{1}{3}\cos\theta \, d\theta =$$

$$= \frac{\pi}{3}\int_0^{\frac{\pi}{2}} \sin 2\theta \, d\theta = -\frac{\pi}{6}\left[\cos 2\theta\right]_0^{\frac{\pi}{2}} =$$

$$= -\frac{\pi}{6}(-1-1) = \frac{\pi}{3}$$