Uppgift 1(a). $D_f = \{(x,y):\ x^2 + (y+\frac{1}{2})^2 > \frac{1}{4}\}$

(b). $f'_x(x,y) = 2 + \frac{2x}{x^2+y^2+y},\ f'_y(x,y) = \frac{2y+1}{x^2+y^2+y},\ f(1,-1) = 2,\ f'_x(1,-1) = 4,\ f'_y(1,-1) = -1$
Tangentplan $z = 2 + 4(x-1) - (y+1) = -3 + 4x - y$ och normallinje $\frac{x-1}{-4} = y + 1 = z - 2$.

(c). $D_{\boldsymbol{u}} f(1,-1) = \boldsymbol{u} \cdot \boldsymbol{\nabla} f(1,-1) = \frac{1}{\sqrt{2}}(1,1)\cdot(4,-1) = \frac{3}{\sqrt{2}}$

Uppgift 2.

$$\begin{cases} u = xy \\ v = \frac{1}{y} \end{cases} \qquad \begin{cases} u'_x = y & u'_y = x \\ v'_x = 0 & v'_y = -\frac{1}{y^2} \end{cases} \qquad \begin{cases} x = uv \\ y = \frac{1}{v} \end{cases}$$

$$\begin{aligned}
&z'_x = z'_u u'_x + z'_v v'_x = y z'_u \\
&z'_y = z'_u u'_y + z'_v v'_y = x z'_u - \tfrac{1}{y^2} z'_v \\
&z''_{xx} = \tfrac{\partial}{\partial x}(y z'_u) = y \tfrac{\partial}{\partial x}(z'_u) = y(z''_{uu} u'_x + z'_{uv} v'_x) = y(z''_{uu} y) = y^2 z''_{uu} \\
&z''_{xy} = \tfrac{\partial}{\partial y}(y z'_u) = z'_u + y \tfrac{\partial}{\partial y}(z'_u) = z'_u + y(z''_{uu} u'_y + z''_{uv} v'_y) = z'_u + y(z''_{uu} x + z''_{uv}(-\tfrac{1}{y^2})) = \\
&= z'_u + xy z''_{uu} - \tfrac{1}{y} z''_{uv} \\
&x z''_{xx} - y z''_{xy} + z'_x - y = z''_{uv} - \tfrac{1}{v} = 0, \text{ dvs. } z''_{uv} = \tfrac{1}{v}.
\end{aligned}$$

Vi har $z'_u = \ln v + f(u)$ och $z = u \ln v + F(u) + h(v) = -xy \ln y + g(xy) + h(\frac{1}{y})$.

Uppgift 3. Skriver x, y istället för x_1, x_2 så att $f(x,y) = \frac{3xy - x^2y^2}{x+y}$.

$$\begin{aligned}
f'_x(x,y) &= \tfrac{(3y-2xy^2)(x+y)-(3xy-x^2y^2)}{(x+y)^2} = \tfrac{3y^2-x^2y^2-2xy^3}{(x+y)^2} = y^2 \tfrac{3-x^2-2xy}{(x+y)^2} = 0 \Leftrightarrow 3 - x^2 - 2xy = 0 \\
f'_y(x,y) &= \tfrac{(3x-2x^2y)(x+y)-(3xy-x^2y^2)}{(x+y)^2} = \tfrac{3x^2-x^2y^2-2x^3y}{(x+y)^2} = x^2 \tfrac{3-y^2-2xy}{(x+y)^2} = 0 \Leftrightarrow 3 - y^2 - 2xy = 0
\end{aligned}$$

Vi ser direkt att $x^2 = y^2$ och därmed $x = y$, eftersom $x > 0, y > 0$. Insättning i en av ekvationerna ger $3 - x^2 - 2x^2 = 0$, dvs. $x^2 = 1$ och därmed $x = 1$, eftersom $x > 0$.

Alltså är $(1,1)$ enda stationära punkten.

$$\begin{aligned}
f''_{xx}(x,y) &= y^2 \tfrac{\partial}{\partial x}\left(\tfrac{3-x^2-2xy}{(x+y)^2}\right) = y^2 \tfrac{(-2x-2y)(x+y)^2-(3-x^2-2xy)2(x+y)}{(x+y)^4} \\
f''_{xy}(x,y) &= 2y\tfrac{3-x^2-2xy}{(x+y)^2} + y^2 \tfrac{\partial}{\partial y}\left(\tfrac{3-x^2-2xy}{(x+y)^2}\right) = 2y\tfrac{3-x^2-2xy}{(x+y)^2} + y^2 \tfrac{-2x(x+y)^2-(3-x^2-2xy)2(x+y)}{(x+y)^4}.
\end{aligned}$$

Vi får $f''_{xx}(1,1) = -1$ och $f''_{yy}(1,1) = -1$, p.g.a. symmetri, samt $f''_{xy}(1,1) = -\frac{1}{2}$.

Detta ger Hesssianen

$$\boldsymbol{H}(1,1) = \begin{bmatrix} -1 & -\frac{1}{2} \\ -\frac{1}{2} & -1 \end{bmatrix} \text{ med } \det(\boldsymbol{H} - \lambda \boldsymbol{I}) = (\lambda+1)^2 - \frac{1}{4} = 0 \Leftrightarrow \lambda_{1,2} = -1 \pm \frac{1}{2}$$

dvs. $\lambda_{1,2}$ båda < 0. Alltså har vi en maxpunkt.

Uppgift 4. $I = \iiint_D z\sqrt{x^2+y^2}\,dV$ över $D = \{(x,y,z):\ x^2+y^2 \le z \le \sqrt{x^2+y^2}\}$.
D begränsas av cylindern $x^2 + y^2 \le 1$ så om vi låter $E = \{(x,y):\ x^2+y^2 \le 1\}$ får vi

$$\begin{aligned}
I &= \iint_E \sqrt{x^2+y^2} \left(\int_{x^2+y^2}^{\sqrt{x^2+y^2}} z\,dz \right) dxdy = \iint_E \sqrt{x^2+y^2} \left[\frac{z^2}{2} \right]_{z=x^2+y^2}^{z=\sqrt{x^2+y^2}} dxdy = \\
&= \frac{1}{2} \iint_E \sqrt{x^2+y^2} \left(x^2+y^2 - (x^2+y^2)^2 \right) dxdy = \{\text{Polära koordinater}\} = \\
&= \frac{1}{2} \int_0^1 \int_0^{2\pi} r(r^2 - r^4) r\,drd\theta = \pi \int_0^1 (r^4 - r^6)\,dr = \pi \frac{2}{35}
\end{aligned}$$

Uppgift 5. Med Green får vi

$$\int_C (x^2 - xy)\,dx + (xy - y^2)\,dy = \iint_D \frac{\partial}{\partial x}(xy - y^2) - \frac{\partial}{\partial y}(x^2 - xy)\,dxdy = \iint_D y + x\,dxdy =$$

$$= \int_0^1 \left(\int_y^{2-y} x + y\,dx \right) dy = \int_0^1 \left[\frac{x^2}{2} + xy \right]_{x=y}^{x=2-y} dy = \int_0^1 2 - 2y^2\,dy = \frac{4}{3}$$

Uppgift 6. $f(\boldsymbol{x}) = x_1^2 - 3x_1 + x_1x_2 + x_2^2,\ \nabla f(\boldsymbol{x}) = (2x_1 - 3 + x_2, x_1 + 2x_2)$

(a). $\boldsymbol{x}_0 = (1,1),\ \nabla f(\boldsymbol{x}_0) = (0,3)$

$\boldsymbol{x}(s) = \boldsymbol{x}_0 - s\boldsymbol{\nabla} f(\boldsymbol{x}_0) = (1,1) - s\,(0,3) = (1, 1-3s)$

$g(s) = f(\boldsymbol{x}(s)) = f(1, 1-3s) = 1 - 3 + (1-3s) + (1-3s)^2 = -9s + 9s^2$

$g'(s) = -9 + 18s = 0 \Leftrightarrow \hat{s} = \frac{1}{2}$

$\boldsymbol{x}_1 = \boldsymbol{x}_0 - \hat{s}\,\boldsymbol{\nabla} f(\boldsymbol{x}_0) = (1,1) - \hat{s}\,(0,3) = (1, -\frac{1}{2})$

(b). Vi har $g'(s) = \frac{d}{ds} f(\boldsymbol{x}(s)) = \boldsymbol{\nabla} f(\boldsymbol{x}(s))^T \boldsymbol{x}'(s) = -\boldsymbol{\nabla} f(\boldsymbol{x}(s))^T \boldsymbol{\nabla} f(\boldsymbol{x}_0)$.

Så om $g'(\hat{s}) = 0$ så är $\boldsymbol{\nabla} f(\boldsymbol{x}(\hat{s}))^T \boldsymbol{\nabla} f(\boldsymbol{x}_0) = \boldsymbol{\nabla} f(\boldsymbol{x}_1) \cdot \boldsymbol{\nabla} f(\boldsymbol{x}_0) = 0$

(För vårt problem ser vi direkt att $\boldsymbol{\nabla} f(\boldsymbol{x}_1) = (-\frac{3}{2}, 0),\ \ \boldsymbol{\nabla} f(\boldsymbol{x}_1) \cdot \boldsymbol{\nabla} f(\boldsymbol{x}_0) = (-\frac{3}{2}, 0) \cdot (0,3) = 0$)

Uppgift 7 och 8. Teori.