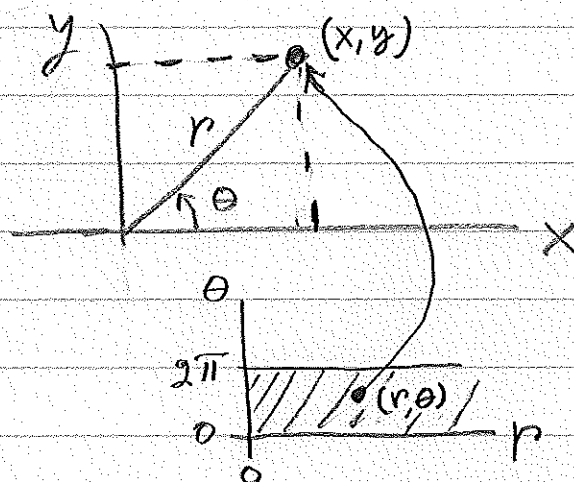


14.4 Polära koordinater

$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \end{cases}$$

$$\begin{cases} r^2 = x^2 + y^2 \\ \tan \theta = y/x \end{cases}$$

$$\begin{cases} r = \sqrt{x^2 + y^2} \\ \theta = \arctan(y/x) \end{cases}$$



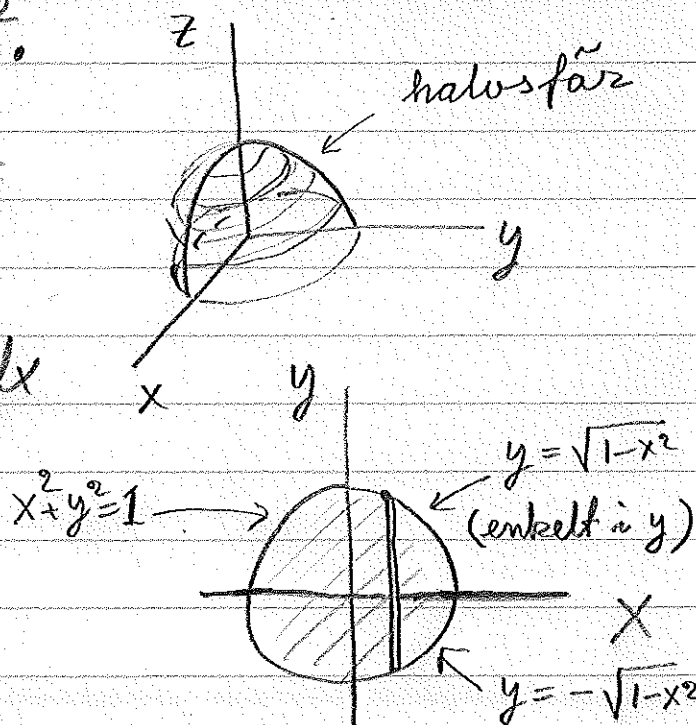
Exempel. Volymen mellan xy -planet

och $z = 1 - x^2 - y^2$.
Volymen under grafen:

$$V = \iint_{x^2 + y^2 \leq 1} (1 - x^2 - y^2) dA =$$

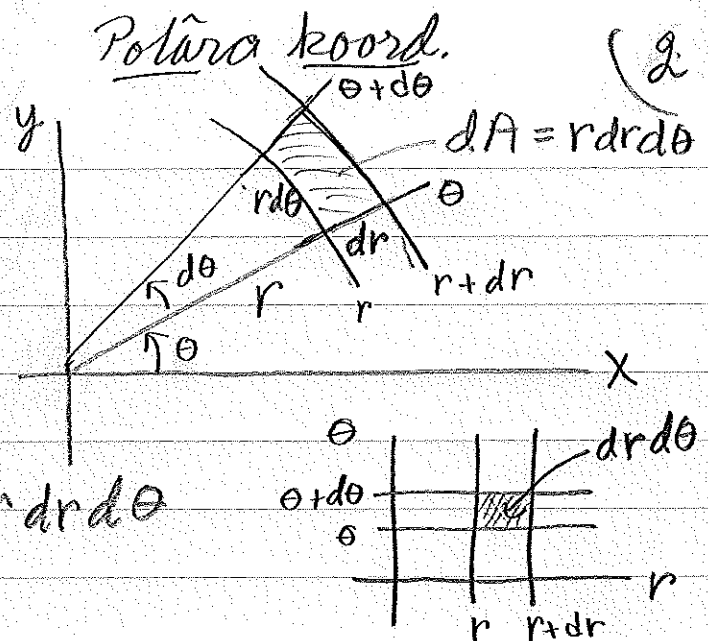
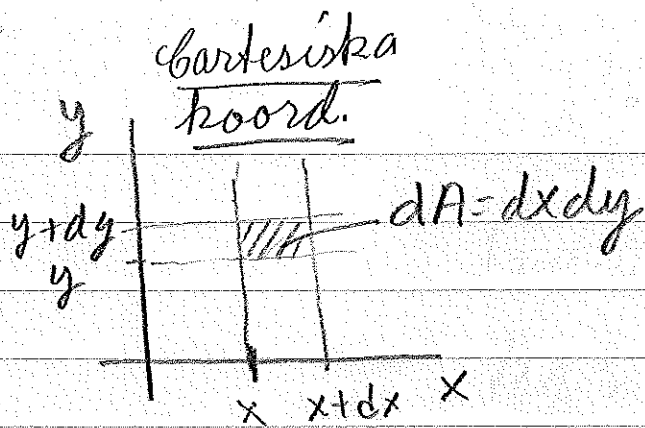
$$= \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (1 - x^2 - y^2) dy dx$$

Mycket svår.

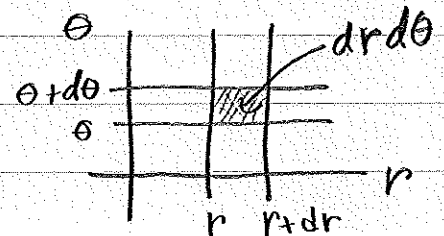


Polära koordinater:

$$V = \iint_{r \leq 1} (1 - r^2) dA$$



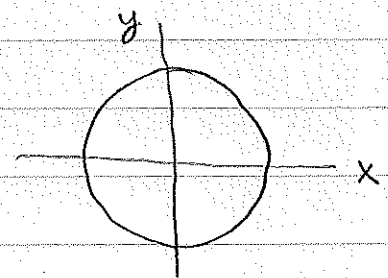
$$dx dy = dA = r dr d\theta$$



$$V = \int_0^{2\pi} \int_0^1 (1-r^2) r dr d\theta =$$

$$= \int_0^{2\pi} \left[\frac{r^2}{2} - \frac{r^4}{4} \right]_{r=0}^{r=1} d\theta$$

$$= \int_0^{2\pi} \left(\frac{1}{2} - \frac{1}{4} \right) d\theta = 2\pi \cdot \frac{1}{4} = \frac{\pi}{2}$$



Exempel 4 Visa att $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$

Bewis. Låt $I = \int_{-\infty}^{\infty} e^{-x^2} dx$. Då blir

$$I^2 = \left(\int_{-\infty}^{\infty} e^{-x^2} dx \right)^2 = \int_{-\infty}^{\infty} e^{-x^2} dx \int_{-\infty}^{\infty} e^{-y^2} dy$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2-y^2} dx dy = \iint_{\mathbb{R}^2} e^{-x^2-y^2} dA$$

$$= \int_0^{2\pi} \int_0^{\infty} e^{-r^2} r dr d\theta = \int_0^{2\pi} d\theta \int_0^{\infty} e^{-r^2} r dr$$

$$= 2\pi \left[-\frac{1}{2} e^{-r^2} \right]_0^\infty = 2\pi \cdot \frac{1}{2} = \pi$$

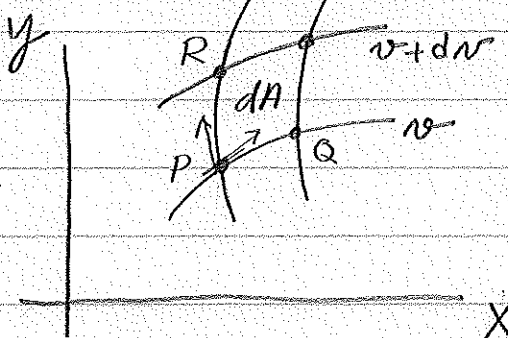
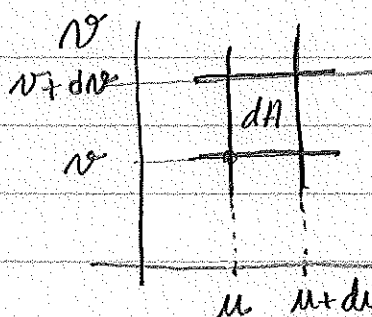
$$\Rightarrow I = \sqrt{\pi}$$

Variabelbyte. Nya koordinater (u, v) .

$$\begin{cases} x = x(u, v) \\ y = y(u, v) \end{cases}$$

$$\begin{cases} u = u(x, y) \\ v = v(x, y) \end{cases}$$

(Gamla koord. x, y
som funktion
av de nya u, v .)



koordinat-
kurvor

Den inversa transformationen finns om
Jacobi-determinanten $\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \neq 0$

$$dA = |\vec{PQ} \times \vec{PR}| \quad (\text{sid 582})$$

$$\vec{PQ} = dx \vec{i} + dy \vec{j} = \{ \text{kedjeregeln} \}$$

$$= \left(\frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv \right) \vec{i} + \left(\frac{\partial y}{\partial u} du + \frac{\partial y}{\partial v} dv \right) \vec{j}$$

$\underbrace{\quad}_{=0} \quad \quad \quad \underbrace{\quad}_{=0}$

$$= \frac{\partial x}{\partial u} du \vec{i} + \frac{\partial y}{\partial u} du \vec{j} = \left(\frac{\partial x}{\partial u} \vec{i} + \frac{\partial y}{\partial u} \vec{j} \right) du$$

$$\vec{PR} = \left(\frac{\partial x}{\partial v} \vec{i} + \frac{\partial y}{\partial v} \vec{j} \right) dv$$

$\underbrace{\quad}_{\text{tangentvektor i P}} \quad \quad \quad \underbrace{\quad}_{\text{tangentvektor i P}}$

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial x}{\partial u} du & \frac{\partial y}{\partial u} du & 0 \\ \frac{\partial x}{\partial v} dv & \frac{\partial y}{\partial v} dv & 0 \end{vmatrix} =$$

$$= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} du dv \vec{k} = \frac{\partial(x,y)}{\partial(u,v)} du dv \vec{k}$$

transponatet
av Jacobi-matrisen

$$\det(A^T) = \det(A)$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{pmatrix} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} \leftarrow \text{determinant}$$

$$dA = |\vec{PQ} \times \vec{PR}| = \left| \frac{\partial(x,y)}{\partial(u,v)} du dv \vec{k} \right| = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du dv$$

längd belopp

$dx dy = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du dv$

dvs

dvs arean skapas med absolutbeloppet av Jacobi-determinanten.

Sats 4 (variabelbyte i dubbelintegral)

$$\iint_D f(x,y) dx dy = \iint_S f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du dv$$



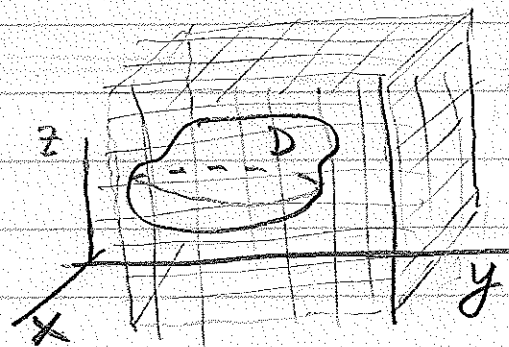
Exempel $\begin{cases} x = r \cos \theta \\ y = r \sin \theta \end{cases}$

$$\frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \cos \theta & \sin \theta \\ -r \sin \theta & r \cos \theta \end{vmatrix} =$$

$$= r \cos^2 \theta + r \sin^2 \theta = r$$

$$dA = |r| dr d\theta = r dr d\theta$$

14.5 Trippelintegralen



$$\iiint_D f(x, y, z) dV = \iiint_D f(x, y, z) dx dy dz$$

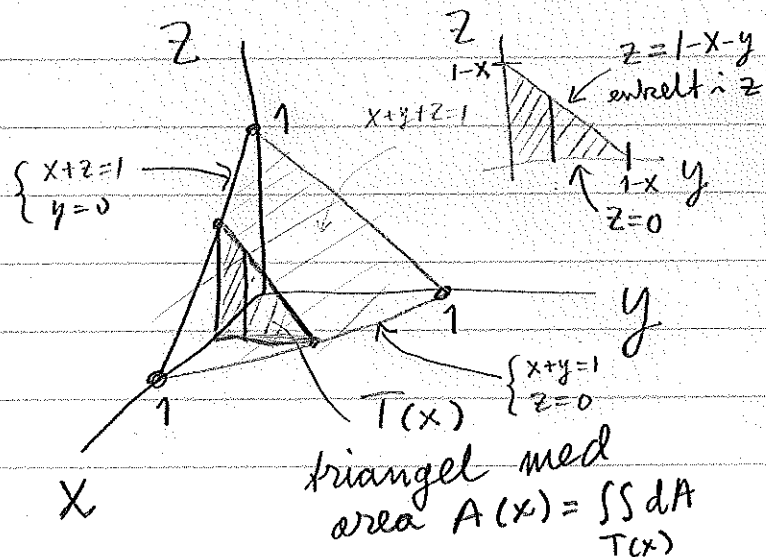
Definieras med partition och Riemann-summa. Kan ibland beräknas med upprepad integration.

Exempel Volymen av tetraeder.

$$V = \iiint_T dV = \int_0^1 A(x) dx$$

$$= \int_0^1 \iint_{T(x)} dA dx$$

$$= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} dz dy dx =$$



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$$= \int_{x=0}^1 \int_{y=0}^{1-x} (1-x-y) dy dx = \int_{x=0}^1 \left[(1-x)y - \frac{1}{2}y^2 \right]_{y=0}^{1-x} dx$$

$$= \int_0^1 \left((1-x)^2 - \frac{1}{2}(1-x)^2 \right) dx$$

$$= \frac{1}{2} \int_0^1 (1-x)^2 dx = \frac{1}{2} \left[-\frac{(1-x)^3}{3} \right]_0^1$$

$$= \frac{1}{6}$$