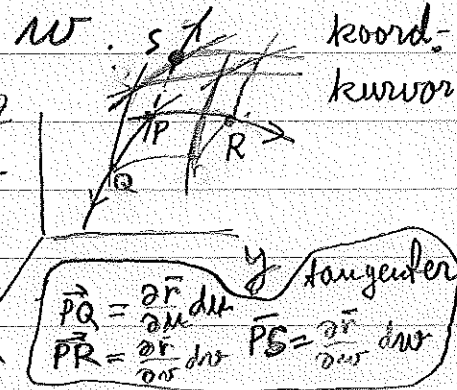
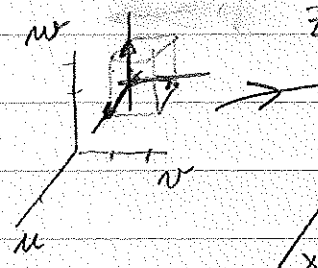


14.6 Variabelbyte i trippelintegraler.Nya koordinater u, v, w . s.t.

$$\begin{cases} x = x(u, v, w) \\ y = y(u, v, w) \\ z = z(u, v, w) \end{cases}$$

Om Jacobi-determinanten är $\neq 0$ så blir volymselementet $dV = |\vec{PQ} \cdot (\vec{PR} \times \vec{PS})|$ sid 582

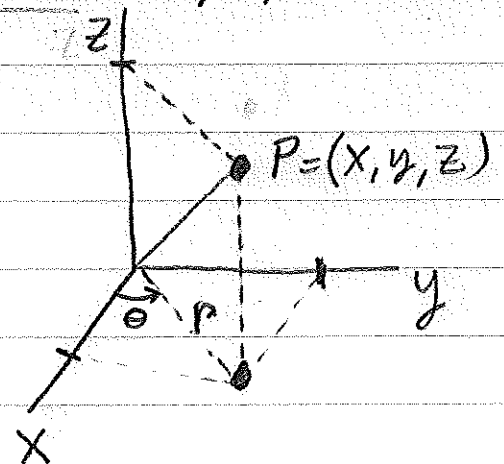
$$dV = dx dy dz = \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| du dv dw$$

(absolutbeloppet av Jacobi-determinanten)

Två speciellt viktiga koordinatsystem:

cylindriska och sfäriska.Cylindriska koordinater: r, θ, z

$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \\ z = z \end{cases}$$



$$0 \leq r < \infty$$

$$0 \leq \theta < 2\pi$$

$$-\infty < z < \infty$$

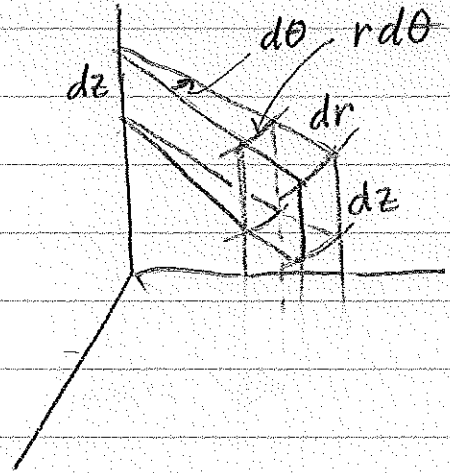
Jacobi-determinanten:

$$\frac{\partial(x, y, z)}{\partial(r, \theta, z)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} =$$

$$= \begin{vmatrix} \cos \theta & -r \sin \theta & 0 \\ \sin \theta & r \cos \theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r \cos^2 \theta + r \sin^2 \theta = r.$$

Volumenelement:

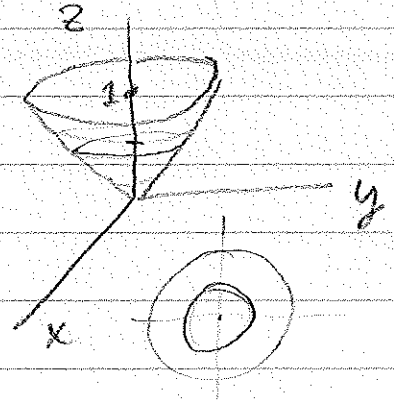
$$dV = r dr d\theta dz$$



Beispielberechnung:

Beispiel Berechnen $I = \iiint_K (x^2 + y^2) dx dy dz$

über K: $\begin{cases} 0 \leq z \leq 1 \\ \sqrt{x^2 + y^2} \leq z \end{cases}$



$$I = \int_{z=0}^1 \int_{\theta=0}^{2\pi} \int_{r=0}^z r^2 r dr d\theta dz$$

$$= \int_0^1 \int_0^z r^3 dr dz \cdot \int_0^{2\pi} d\theta = \int_0^1 \frac{z^4}{4} dz \cdot 2\pi = 2\pi \cdot \frac{1}{20}$$

Sfäriska koordinater

(ρ, ϕ, θ)

(3)

$$\begin{cases} x = \rho \sin \phi \cos \theta \\ y = \rho \sin \phi \sin \theta \\ z = \rho \cos \phi \end{cases}$$

$$\rho = \sqrt{x^2 + y^2 + z^2}$$

$$r = \rho \sin \phi$$

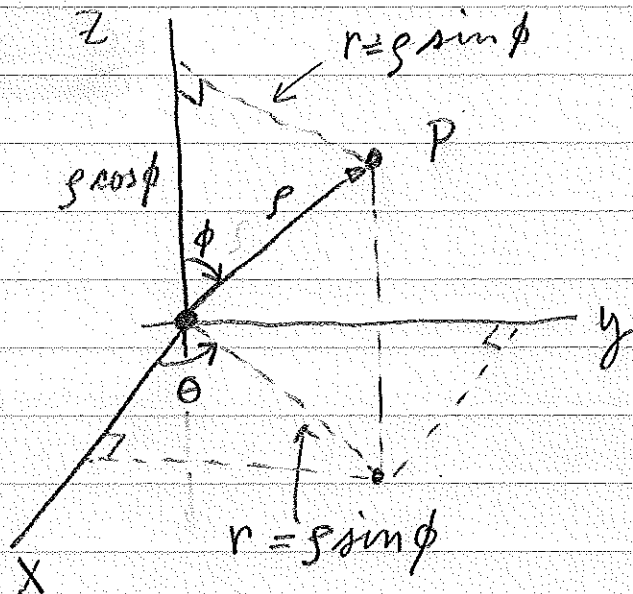
$$\tan \phi = \frac{r}{z} = \frac{\sqrt{x^2 + y^2}}{z}$$

$$\tan \theta = \frac{y}{x}$$

är en radie mot,

$\phi = 0$ nordpolen,

$\phi = \pi/2$ är ekvatorn



$$0 \leq r < \infty$$

$$0 \leq \phi \leq \pi$$

$$0 \leq \theta < 2\pi$$

är en radie mot,

$\phi = \pi$ sydpolen

Jacobi-determinanten:

$$\frac{\partial(x, y, z)}{\partial(\rho, \phi, \theta)} = \begin{vmatrix} \sin \phi \cos \theta & \rho \cos \phi \cos \theta & -\rho \sin \phi \sin \theta \\ \sin \phi \sin \theta & \rho \cos \phi \sin \theta & \rho \sin \phi \cos \theta \\ \cos \phi & -\rho \sin \phi & 0 \end{vmatrix}$$

$$= \sin \phi \cos \theta \cdot \rho^2 \sin^2 \phi \cos \theta + \rho \cos \phi \cos \theta \rho \sin \phi \cos \phi \cos \theta - \rho \sin \phi \sin \theta (-\rho \sin^2 \phi \sin \theta - \rho \cos^2 \phi \sin \theta)$$

$$= \{\text{trig. ettan}\} = \rho^2 \sin \phi$$

$$dV = r^2 \sin \phi \, dr \, d\phi \, d\theta \quad (\text{obs: } \sin \phi \geq 0)$$

(4)

Exempel. Tröghetsmomentet för enhetsklotet m. a. p. z-axeln: $\left(\begin{array}{l} \delta = \text{massfäthet} \\ \left[\frac{\text{kg}}{\text{m}^3} \right] \end{array} \right)$

$$I = \iiint_B (x^2 + y^2) \delta \, dV = \delta \iiint_0^{2\pi} \iiint_0^{\pi} \int_0^1 (r \sin \phi)^2 r^2 \sin \phi \, dr \, d\phi \, d\theta$$

$$= \delta \int_0^1 r^4 \, dr \int_0^{\pi} \sin^3 \phi \, d\phi \int_0^{2\pi} d\theta = \left\{ \begin{array}{l} u = -\cos \phi \\ du = +\sin \phi \, d\phi \\ \sin^2 \phi = 1 - u^2 \\ \phi = 0 \Rightarrow u = -1, \phi = \pi \Rightarrow u = 1 \end{array} \right\}$$

$$= \delta \frac{1}{5} \cdot \int_{-1}^1 (1 - u^2) \, du \cdot 2\pi = \delta \frac{1}{5} \cdot \frac{4}{3} \cdot 2\pi = \frac{8\pi}{15} \delta$$

$$\delta = \text{massfäthet} = \text{konstant} \left[\frac{\text{kg}}{\text{m}^3} \right]$$

14.7 Endast "Moments and Centres of Mass." (5)
sid 798-801

Masscentrum: $(\bar{x}, \bar{y}, \bar{z})$ där

$$\bar{x} = \frac{\iiint_R x \delta(x, y, z) dV}{\iiint_R \delta(x, y, z) dV}$$

δ = massfäthet =
= densitet
[kg/m³]

OSN.

På vektorform: $\bar{r} = \frac{\iiint_R r \delta dV}{\iiint_R \delta dV} = \frac{\iiint_R r dm}{\iiint_R dm}$

Tröghetsmoment m.a.p. axel:

$$I = \iiint_R D^2 \delta(x, y, z) dV = \iiint_R D^2 dm$$

D = avståndet till axeln

