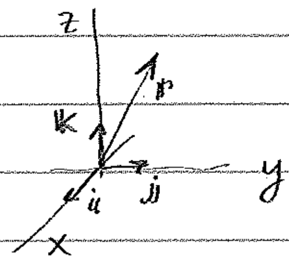
MVE255 2013-14 FÖ 4.2

Idag: FEM-1 forts.
Parametriserad kurva 3.1-3.2
Kurvintegral 3.3

Geometriska beteckningar

I fortsättningen studerar vi kurvor, ytor, vektorfält i rummet och planet. Då använder jag geometriska beteckningar med fetstil istället för matrisbeteckningar.

ex. $\mathbf{r} = r_1\mathbf{i} + r_2\mathbf{j} + r_3\mathbf{k} = (r_1, r_2, r_3)$



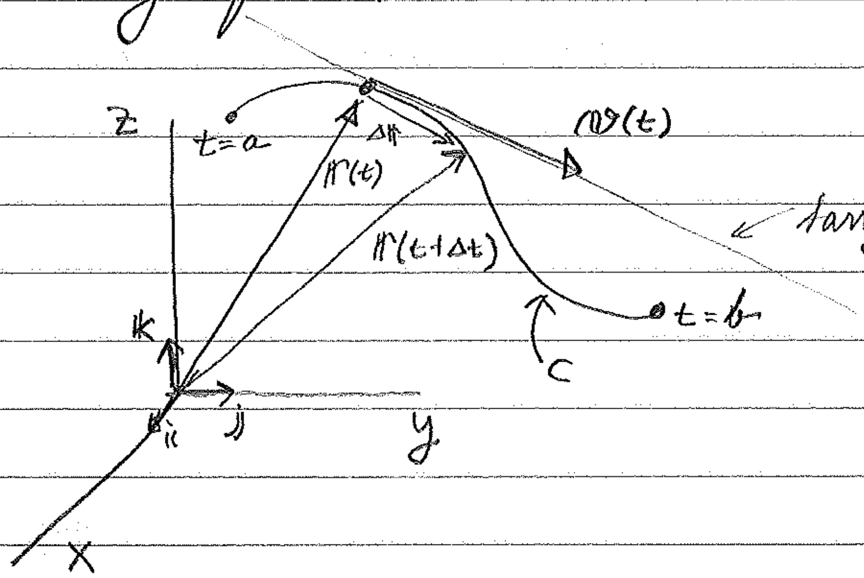
Parametriserad kurva i rummet.

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}, \quad t \in I = [a, b]$$
$$= (x(t), y(t), z(t))$$

$$\begin{cases} x = x(t), \\ y = y(t), \quad t \in I. \\ z = z(t), \end{cases}$$

Geometrisk tolkning: läget av partikel vid tiden t.

Antag: x, y, z kontinuerliga så att partikeln rör sig på kurva C.



Medelhastighet: $\dfrac{\Delta \mathbf{r}}{\Delta t}$

Hastighet (velocity) $\mathbf{v}(t) = \lim\limits_{\Delta t \to 0} \dfrac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t} =$

$$= \lim_{\Delta t \to 0} \frac{\Delta \mathbf{r}}{\Delta t} = \mathbf{r}'(t) = \dot{\mathbf{r}}(t)$$

Fart (speed) $v(t) = |\mathbf{v}(t)|$

Derivera komponentvis: $\mathbf{v} = x'\mathbf{i} + y'\mathbf{j} + z'\mathbf{k}$

Figuren: $\mathbf{v}(t)$ är tangent till kurvan.

<u>Acceleration</u> $\mathbf{a}(t) = \frac{d}{dt}\mathbf{v}(t) = \mathbf{v}'(t)$

$$= \dot{\mathbf{v}} = \frac{d^2}{dt^2}\mathbf{r}(t) = \mathbf{r}''(t) = \ddot{\mathbf{r}}(t)$$

Obs: i boken definieras $a = \frac{d}{dt}|\mathbf{v}| = v'$ inte $a = |\mathbf{a}|$.
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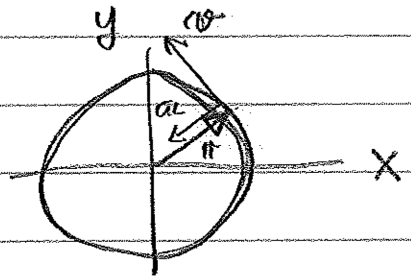
<u>Exempel</u> (cirkel)

$$\mathbf{r} = \big(a\cos(\omega t),\ a\sin(\omega t),\ 0\big), \quad a > 0,\ \omega > 0$$

$$\begin{cases} x = a\cos(\omega t) \\ y = a\sin(\omega t) \\ z = 0 \end{cases}$$

$$x^2 + y^2 = a^2$$



$$\mathbf{v} = \big(-a\omega\sin(\omega t),\ a\omega\cos(\omega t),\ 0\big)$$

$$v = \sqrt{(-a\omega\sin(\omega t))^2 + (a\omega\cos(\omega t))^2}$$

$$= \sqrt{a^2\omega^2} = |a\omega| = a\omega \qquad \text{om } \omega > 0$$

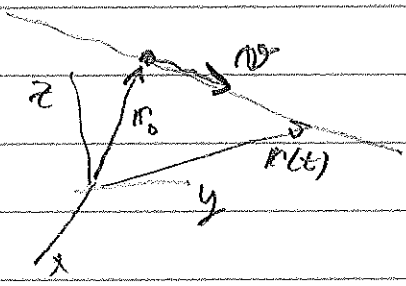
$$\mathbf{r}\bullet\mathbf{v} = \ldots = 0$$

$$\mathbf{a} = \big(-a\omega^2\cos(\omega t),\ -a\omega^2\cos(\omega t),\ 0\big) = -\omega^2\mathbf{r}$$

<u>Exempel</u> (rät linje)

$$\mathbf{r} = \mathbf{r}_0 + t\,\mathbf{v}$$

$$\begin{cases} x = x_0 + t v_1 \\ y = y_0 + t v_2 \\ z = z_0 + t v_3 \end{cases} \qquad \begin{aligned} \mathbf{r}' &= \mathbf{v} \\ \mathbf{r}'' &= 0 \end{aligned}$$



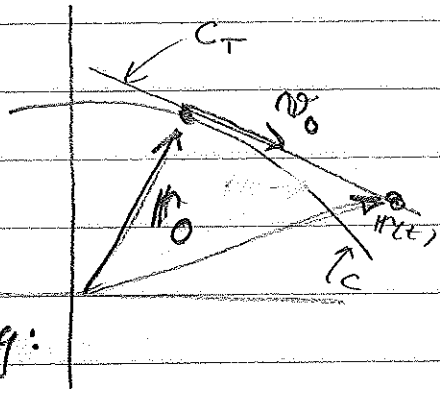
Tangentlinjens ekv.

Välj punkt på kurvan: $\mathbf{r}_0 = \mathbf{r}(t_0)$

C: $\mathbf{r} = \mathbf{r}(t)$, $\mathbf{v}_0 = \mathbf{r}'(t_0)$

Tangentlinjen:

C_T: $\mathbf{r} = \mathbf{r}_0 + t\,\mathbf{v}_0$



eller med linjärisering:

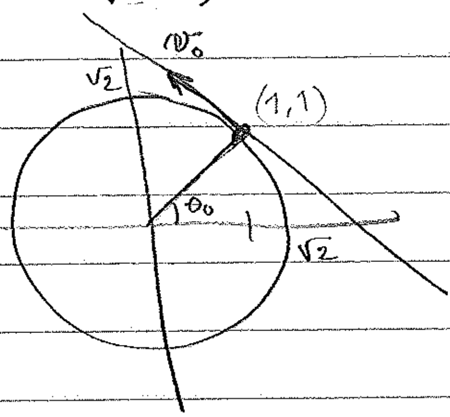
$$C_T: \mathbf{r} = \mathbf{r}(t_0) + (t - t_0)\,\mathbf{r}'(t_0)$$

Inte samma t.

Exempel cirkeln, punkt $(1,1)$

$$\begin{cases} x = \sqrt{2}\cos\theta \\ y = \sqrt{2}\cos\theta \end{cases}$$

$\theta_0 = \pi/4$, $\mathbf{v}_0 = \mathbf{r}'(\pi/4) = (-\sqrt{2}\cos\theta_0, \sqrt{2}\sin\theta_0)$
$= (-1, 1)$



Tangenten: $\mathbf{r} = (1,1) + t(-1,1)$

eller $\begin{cases} x = 1 - t \\ y = 1 + t \end{cases}$

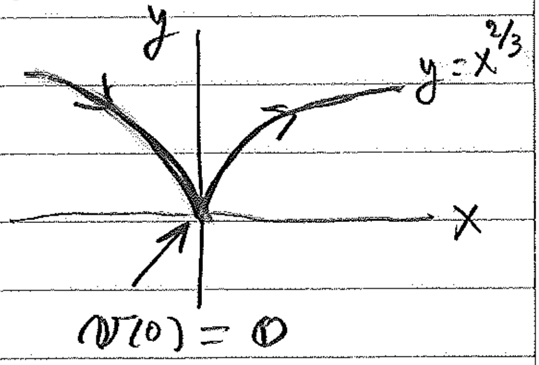
Exempel (spets)

$$\begin{cases} x = t^3 \\ y = t^2 \\ z = 0 \end{cases}$$

$v = (3t^2, 2t, 0)$

Eliminera t: $t = x^{1/3}$, $y = x^{2/3}$

Kurvan är en graf:



Exempel (graf i planet)

$$y = f(x)$$

Välj $t = x$.

$$\begin{cases} x = t \\ y = f(t) \\ z = 0 \end{cases} \qquad v = (1, f'(t), 0)$$

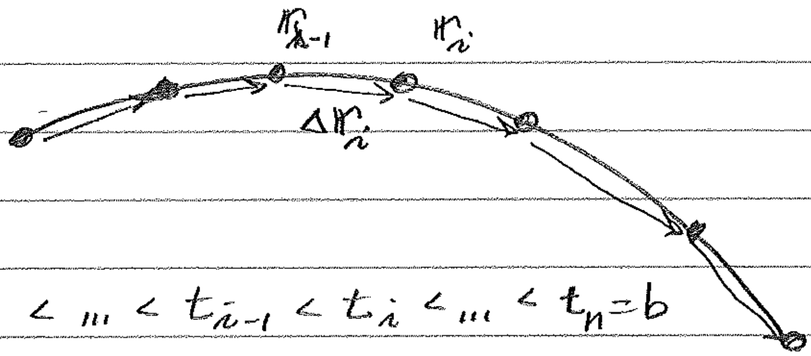
$$|v| = \sqrt{1 + f'(t)^2} = \sqrt{1 + f'(x)^2}$$

Sats 3.1.4 Räkneregler

t.ex. $\left(\mathbb{r}_1(t) \cdot \mathbb{r}_2(t)\right)' = \mathbb{r}_1'(t) \cdot \mathbb{r}_2(t) + \mathbb{r}_1(t) \cdot \mathbb{r}_2'(t)$

$$\frac{d}{dt}|\mathbb{r}(t)|^2 = \frac{d}{dt}\, \mathbb{r}(t) \cdot \mathbb{r}(t) = 2\, \mathbb{r}(t) \cdot \mathbb{r}'(t)$$

Båglängd



Nät: $a = t_0 < t_1 < \dots < t_{i-1} < t_i < \dots < t_n = b$

$\Delta t_i = t_i - t_{i-1}$

$\mathbf{r}_i = \mathbf{r}(t_i)$, $\Delta \mathbf{r}_i = \mathbf{r}_i - \mathbf{r}_{i-1}$

Approximativ längd:

$$s_n = \sum_{i=1}^{n} |\Delta \mathbf{r}_i| = \sum_{i=1}^{n} \left| \frac{\Delta \mathbf{r}_i}{\Delta t_i} \right| \Delta t_i$$

Längd: $\max(\Delta t_i) \to 0$

$$L = \int_a^b \left| \frac{d\mathbf{r}}{dt} \right| dt = \int_a^b |\mathbf{r}'(t)| \, dt$$

Båglängd: $s(t) = \int_a^t |\mathbf{r}'(u)| \, du$

Derivera: $s'(t) = |\mathbf{r}'(t)| = v(t)$

Båglängdselementet

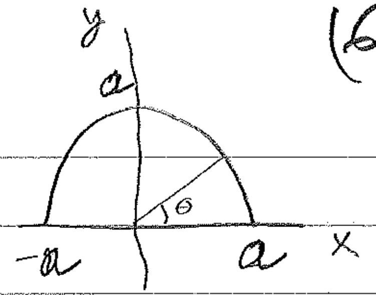
$$ds = v(t) \, dt = |\mathbf{r}'(t)| \, dt$$

Längd: $L = \int_C ds = \int_a^b |\mathbf{r}'(t)| \, dt$

Båglängden beror ej på vilken parameter vi valt.

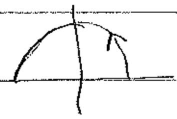
Exempel (halvcirkeln)

$x^2 + y^2 = a^2$



1)

$$\begin{cases} x = a\cos\theta \\ y = a\sin\theta \end{cases}, \quad \theta \in [0, \pi]$$

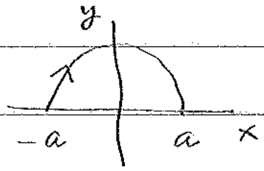


$\mathbf{r}'(\theta) = (-a\sin\theta,\ a\cos\theta)$, $|\mathbf{r}'(\theta)| = a$

$ds = a\,d\theta$

$L = \int_C ds = \int_0^\pi |\mathbf{r}'(\theta)|\,d\theta = \int_0^\pi a\,d\theta = a\pi$

2) $t = x$ motsatt riktning!



$$\begin{cases} x = t \\ y = \sqrt{a^2 - t^2} \end{cases}, \quad t \in [-a, a]$$

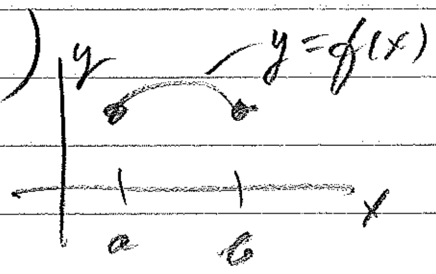
$\mathbf{r}'(t) = \left(1, \frac{-t}{\sqrt{a^2-t^2}}\right)$, $|\mathbf{r}'(t)| = \sqrt{1^2 + \left(\frac{t}{\sqrt{a^2-t^2}}\right)^2} =$

$= \sqrt{\frac{a^2 - t^2 + t^2}{a^2 - t^2}} = \frac{a}{\sqrt{a^2-t^2}}$

$ds = \frac{a}{\sqrt{a^2-t^2}}\,dt$

$L = \int_C ds = \int_{-a}^{a} \frac{a}{\sqrt{a^2-t^2}}\,dt = \ldots = a\pi$

Exempel (graf i planet)



$y = f(x)$, $x \in [a, b]$

$ds = \sqrt{1 + f'(x)^2}\,dx$

Kurvintegral av skalärfält.

$f : \mathbb{R}^3 \to \mathbb{R}$ skalärt fält

$C: \mathbf{r} = \mathbf{r}(t), \quad t \in [a,b]$ kurva

Kurvintegral:

$$\int_C f \, ds = \int_a^b f(\mathbf{r}(t)) \, |\mathbf{r}'(t)| \, dt$$

Läs Holsrud 3.1, 3.2, 3.3