MVE 255 2013-14 FÖ 6.1

Idag: 6.7 Variabelbyte
6.8 Generaliserad integral

## Variabelbyte

Nya koordinater $(u, v)$.

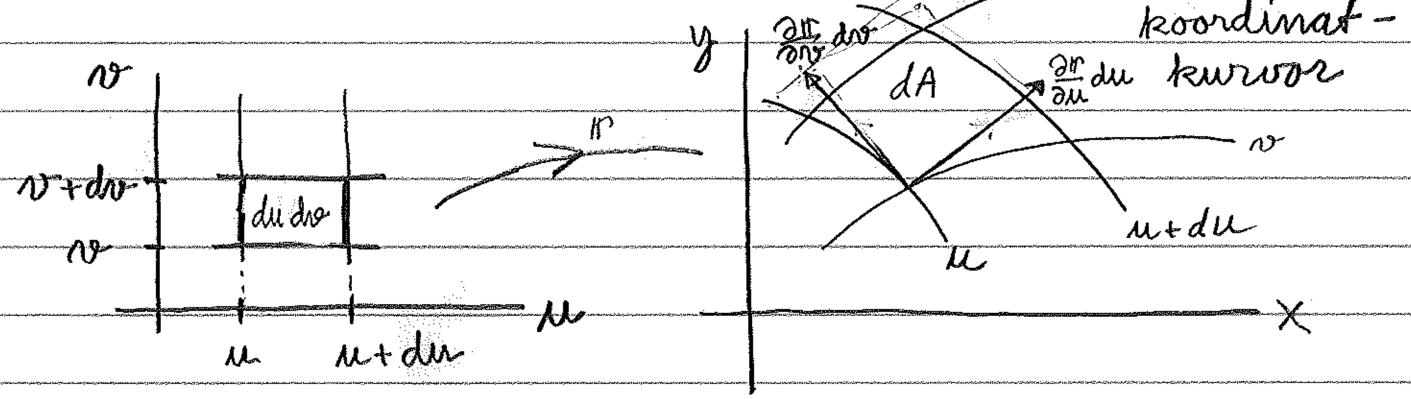
$\mathbf{r} = \mathbf{r}(u, v)$

Transformation:

$$\begin{cases} x = x(u,v) \\ y = y(u,v) \end{cases}$$

Invers transformation:

$$\begin{cases} u = u(x,y) \\ v = v(x,y) \end{cases}$$




$$dA = \left| \frac{\partial \mathbf{r}}{\partial u} du \times \frac{\partial \mathbf{r}}{\partial v} dv \right| \quad \text{areaelementet}$$

spänns upp av tangenterna $\frac{\partial \mathbf{r}}{\partial u} du$ och $\frac{\partial \mathbf{r}}{\partial v} dv$.

$$\frac{\partial \mathbf{r}}{\partial u} du \times \frac{\partial \mathbf{r}}{\partial v} dv = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial x}{\partial u} du & \frac{\partial y}{\partial u} du & 0 \\ \frac{\partial x}{\partial v} dv & \frac{\partial y}{\partial v} dv & 0 \end{vmatrix} = \left\{ \begin{matrix} \text{bryt ut} \\ du \text{ och } dv \end{matrix} \right\}$$

$$= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} du\, dv\, \mathbf{k} = \frac{\partial(x,y)}{\partial(u,v)} du\, dv\, \mathbf{k}$$

# Jacobideterminanten för transformationen:

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\left(\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{bmatrix}\right) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} \overset{\det(A^T)=\det(A)}{=}$$

= transponatet $\mathbb{r}'(u,v)^T$ av Jacobimatrisen

$$= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \det(\mathbb{r}'(u,v))$$

determinant-streck = Jacobimatrisen $\mathbb{r}'(u,v)$

Vi får nu

$$dA = \left|\frac{\partial \mathbb{r}}{\partial u}\,du \times \frac{\partial \mathbb{r}}{\partial v}\,dv\right| = \left|\frac{\partial(x,y)}{\partial(u,v)}\,du\,dv\,\mathbb{k}\right| =$$

längd

abs. belopp

$$= \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

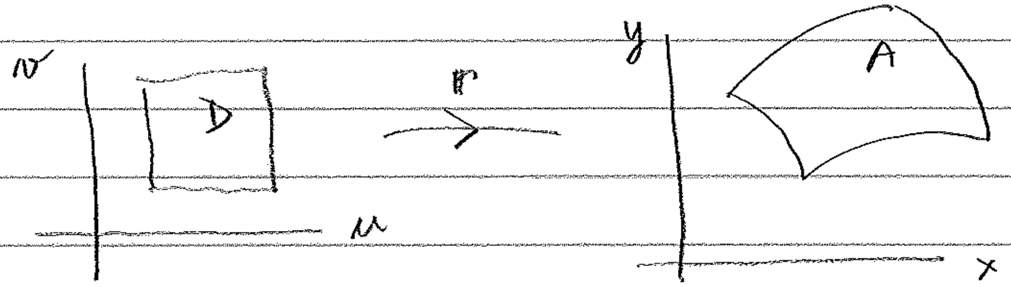
dvs

$$\boxed{\begin{aligned} dx\,dy &= \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv \\ &= \left|\det(\mathbb{r}'(u,v))\right| du\,dv \end{aligned}}$$

Obs: $\left|\frac{\partial(x,y)}{\partial(u,v)}\right|$ är <u>yt-skalan</u> vid transformationen.

## Sats 6.7.1 (Variabelbyte i dubbelintegral)

$$\iint_A f(x,y)\,dx\,dy = \iint_D f(\mathbf{r}(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$




## Exempel Polära koord $(r,\theta)$.

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

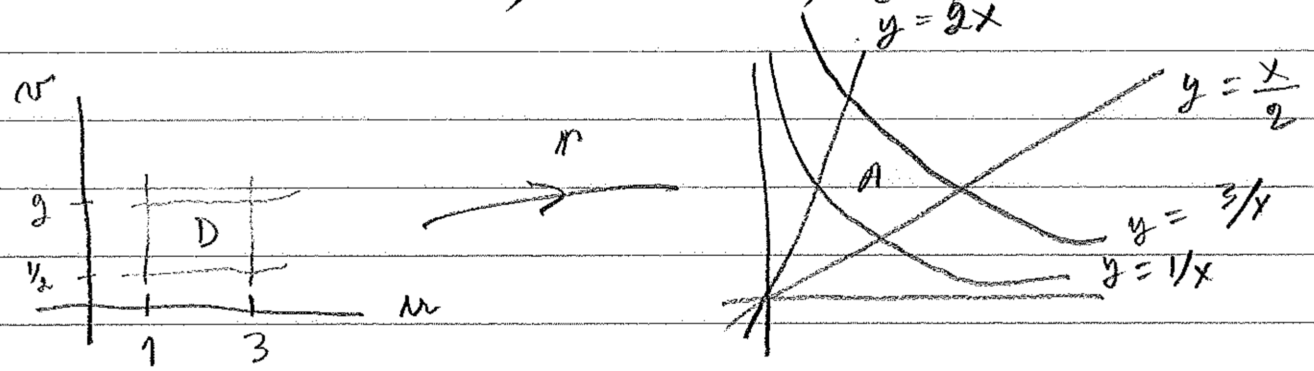
$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = r$$

$$dA = \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| dr\,d\theta = |r|\,dr\,d\theta = r\,dr\,d\theta.$$

## Exempel 2 sid 478

Beräkna $\iint_A \frac{x}{y}\,dx\,dy$ där $A$ är området mellan kurvorna

$$y = \frac{1}{x}, \quad y = \frac{3}{x}, \quad y = \frac{x}{2}, \quad y = 2x.$$




Vi har $1 \leq xy \leq 3, \quad \frac{1}{2} \leq \frac{y}{x} \leq 2$.

Välj $u = xy, \quad v = y/x$. Detta är inversa transformationen. Lös ut $x, y$.

Ekv (2): $y = vx$. I ekv (1): $u = vx^2, \quad x = (\pm)\sqrt{\frac{u}{v}}$

Sedan $y = v\sqrt{\frac{u}{v}} = \sqrt{uv}$.

Transformationen blir

$$\begin{cases} x = \sqrt{\frac{u}{v}} \\ y = \sqrt{uv} \end{cases}$$

Jacobideterminanten: $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} =$

$$= \begin{vmatrix} \frac{1}{2\sqrt{uv}} & -\frac{\sqrt{u}}{2\sqrt{v^3}} \\ \frac{\sqrt{v}}{2\sqrt{u}} & \frac{\sqrt{u}}{2\sqrt{v}} \end{vmatrix} = \frac{1}{2v}$$

Integralen blir

$$\iint_A \frac{x}{y}\,dx\,dy = \iint_D \frac{\sqrt{\frac{u}{v}}}{\sqrt{uv}} \frac{1}{2v}\,du\,dv =$$

(fel i boken)

$$= \iint_D \frac{1}{2v^2}\,du\,dv = \int_1^3 \left( \int_{1/2}^2 \frac{1}{2v^2}\,dv \right) du$$

$$= \int_1^3 du \int_{1/2}^2 \frac{1}{2v^2}\,dv = 2 \cdot \left[ \frac{1}{-2v} \right]_{1/2}^2$$

$$= 2\left( \frac{1}{-4} - \frac{1}{-1} \right) = 2 \cdot \frac{3}{4} = \frac{3}{2}$$

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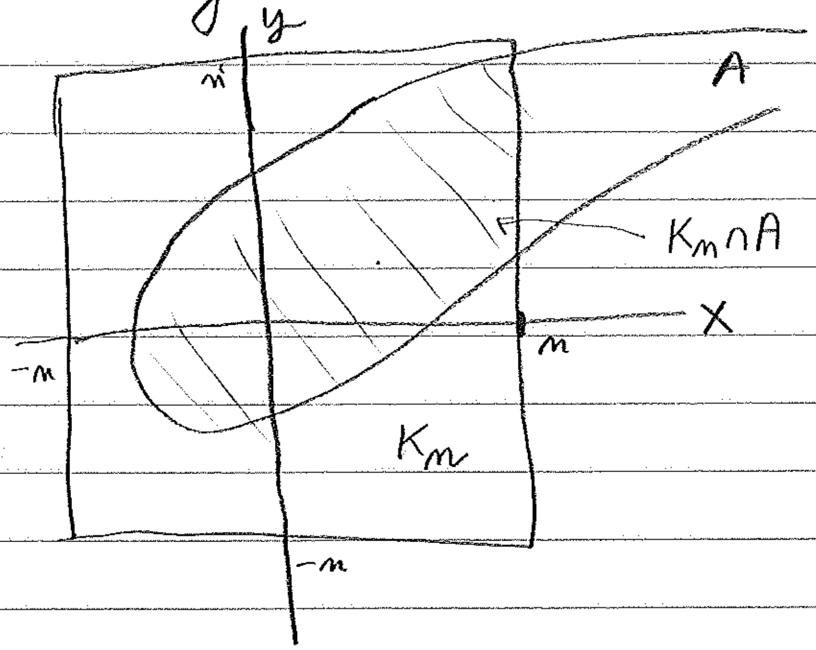
Hoppa över sid 481–488.

# 6.8 Generaliserad integral

Obegränsat område A:




Icke-negativ integrand $f(x,y) \geq 0$.

<u>Definition</u> Om $f: A \to \mathbb{R}$ är kontinuerlig och icke-negativ så definierar vi

$$\iint_A f(x,y)\,dx\,dy = \lim_{n\to\infty} \iint_{K_n \cap A} f(x,y)\,dx\,dy$$

om gränsvärdet existerar. I så fall säger vi att integralen <u>konvergerar</u>. Annars är den <u>divergent</u>.

Obs: eftersom $f(x,y) \geq 0$ finns det bara två möjligheter:

$\iint_A f\, dA$ konvergerar

eller divergerar mot $+\infty$.

Exempel Rektanglar $K_n$:

$$\iint_{\mathbb{R}^2} e^{-\frac{(x^2+y^2)}{2}} dx\,dy = \lim_{n\to\infty} \iint_{K_n} e^{-\frac{(x^2+y^2)}{2}} dx\,dy =$$

$$= \lim_{n\to\infty} \int_{-n}^{n} \left( \int_{-n}^{n} e^{-\frac{x^2}{2}-\frac{y^2}{2}} dx \right) dy =$$

$$= \lim_{n\to\infty} \int_{-n}^{n} e^{-y^2/2} dy \int_{-n}^{n} e^{-\frac{x^2}{2}} dx$$

$$= \left( \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx \right)^2 = ?$$ vad blir det.

Försök igen. Diskar $B(0,n)$:

$$\iint_{\mathbb{R}^2} e^{-\frac{(x^2+y^2)}{2}} dx\,dy = \lim_{n\to\infty} \iint_{B(0,n)} e^{-\frac{(x^2+y^2)}{2}} dx\,dy =$$

$$= \{\text{polära}\} = \lim_{n\to\infty} \int_0^{2\pi} \int_0^{n} e^{-\frac{r^2}{2}} r\, dr\, d\theta$$

$$= \lim_{n\to\infty} 2\pi \left[ e^{-\frac{r^2}{2}} \right]_0^n = \lim_{n\to\infty} 2\pi \left(1 - e^{-\frac{n^2}{2}}\right)$$

$$= 2\pi$$

Alltså: $\displaystyle\iint_{\mathbb{R}^2} e^{-\frac{(x^2+y^2)}{2}}\,dxdy = 2\pi$

Dessutom: $\displaystyle\int_{-\infty}^{\infty} e^{-x^2/2}\,dx = \sqrt{2\pi}$

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Om $f$ byter tecken, dela upp

$$f(x) = f_+(x) - f_-(x)$$

där
$$f_+(x) = \begin{cases} f(x) & \text{om } f(x) > 0 \\ 0 & \text{annars} \end{cases}$$

$$f_-(x) = \begin{cases} -f(x) & \text{om } f(x) < 0 \\ 0 & \text{annars} \end{cases}$$

$\iint_A f\,dA$ är konvergent om båda $\iint_A f_\pm\,dA$ är konvergenta.