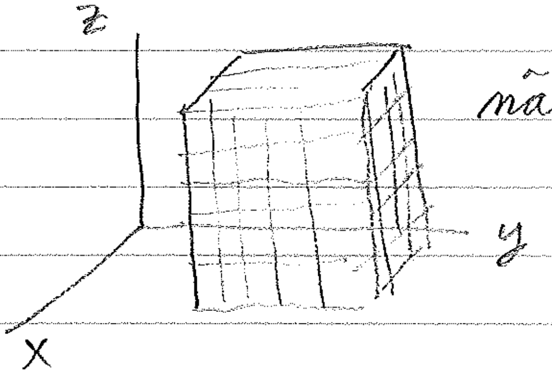
MVE255 2013-14 FÖ 6.2

Idag: 6.9 Trippelintegraler
6.10 Variabelbyte
~~6.18 Ytintegraler~~

6.9 Trippelintegralen

$$\iiint_R f(x,y,z)\,dx\,dy\,dz = \lim \sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K} f(x_{ijk})\,\Delta x_i\,\Delta y_j\,\Delta z_k$$

R rektangulärt område i $\mathbb{R}^3$.



Allmänt begränsat område D: integrera över stort rekt. område R så att $D \subseteq R$, sätt f till 0 utanför D.

Kan ibland räknas ut med upprepad integration.

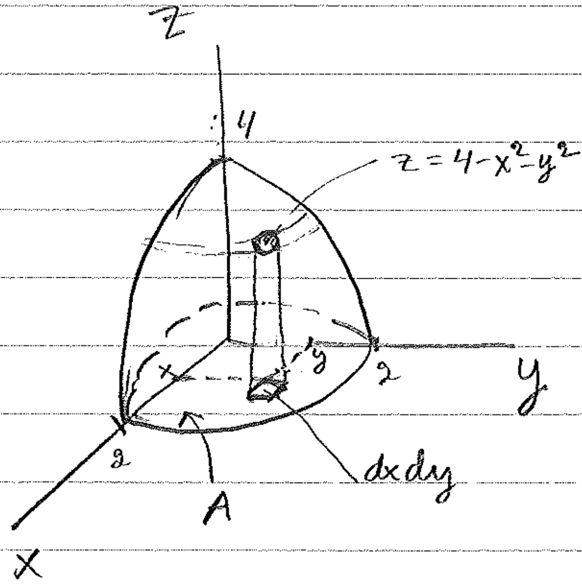
Rektangulärt: $R = [a_1, a_2] \times [b_1, b_2] \times [c_1, c_2]$

$$\iiint_R f(x,y,z)\,dx\,dy\,dz = \int_{a_1}^{a_2}\left(\int_{b_1}^{b_2}\left(\int_{c_1}^{c_2} f(x,y,z)\,dz\right)dy\right)dx$$

Allmänt område är oftast svårt.

Exempel 2

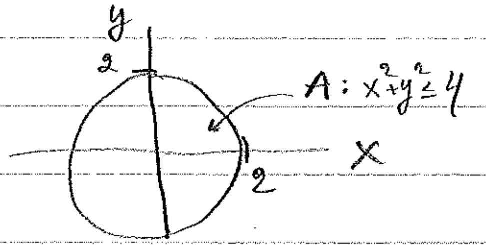
$$I = \iiint_D x \, dx\, dy\, dz$$



$D: \quad 0 \le z \le 4 - x^2 - y^2$

Enkelt i z, dvs
mellan två grafer
$z = 0$ och $z = 4 - x^2 - y^2$.

Bottenytan



$$I = \iint_A \left(\int_0^{4-x^2-y^2} x \, dz \right) dx\, dy$$

$$= \iint_A \left(x \left[z \right]_0^{4-x^2-y^2} \right) dx\, dy$$

$$= \iint_A x(4 - x^2 - y^2)\, dx\, dy = \{ \text{poläre} \} =$$

$$= \int_0^{2\pi}\!\!\left(\int_0^{2} r\cos\theta\,(4 - r^2)\, r\, dr\right) d\theta$$

$$= \int_0^{2\pi} \cos\theta \, d\theta \int_0^2 r^2(4 - r^2)\, dr = \underbrace{\left[\sin\theta \right]_0^{2\pi}}_{=0} \int_0^2 \cdots \, dr = 0$$

Dum uträkning i boken.

(2b

I Matlab:

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>> f = @(x,y,z)(X.*(Z <= 4-X.^2-y.^2))
>> I = triplequad(f,-2,2,-2,2,0,4)
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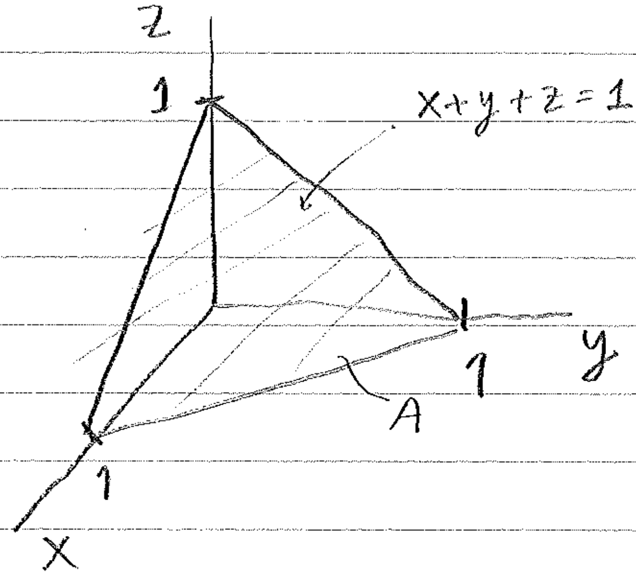
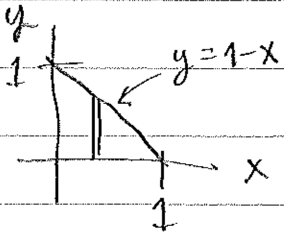
$$\underbrace{-2,2,-2,2,0,4}_{[-2,2]\times[-2,2]\times[0,4]}$$

Exempel Volymen av tetraeder.

T begränsas av planet $x+y+z=1$ och av koord. planen $x=0$, $y=0$ och $z=0$.



Bottenytan:



Enkelt i z: $0 \leq z \leq 1-x-y$

$$V = \iiint_T dx\,dy\,dz = \iint_A \left(\int_0^{1-x-y} dz \right) dx\,dy$$

$$= \int_0^1 \left(\int_0^{1-x} \left(\int_0^{1-x-y} dz \right) dy \right) dx$$

$$= \int_0^1 \int_0^{1-x} (1-x-y)\, dy \quad dx$$

$$= \int_0^1 \left[(1-x)y - \frac{1}{2}y^2 \right]_0^{1-x} dx$$

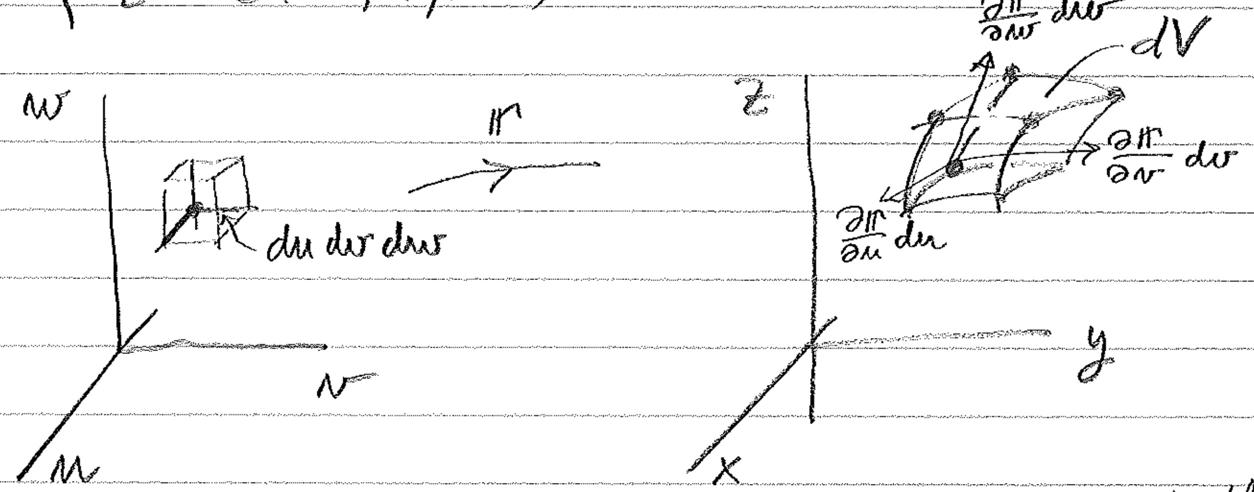
$$= \int_0^1 \left((1-x)^2 - \frac{1}{2}(1-x)^2 \right) dx$$

$$= \frac{1}{2} \int_0^1 (1-x)^2\, dx = \frac{1}{2}\left[-\frac{1}{3}(1-x)^3 \right]_0^1 = \frac{1}{6}.$$

6.10 Variabelbyte.

Nya koordinater (u, v, w).

$$\begin{cases} x = x(u,v,w) \\ y = y(u,v,w) \\ z = z(u,v,w) \end{cases} \qquad \mathbf{r} = \mathbf{r}(u,v,w)$$



dV spänns upp av tangenterna.

$$dV = \left| \frac{\partial \mathbf{r}}{\partial u} du \cdot \left(\frac{\partial \mathbf{r}}{\partial v} dv \times \frac{\partial \mathbf{r}}{\partial w} dw \right) \right| =$$

absolutbelopp av trippelprodukten

$$= \left| \frac{\partial (x,y,z)}{\partial (u,v,w)} \right| du\, dv\, dw =$$

$$= \underbrace{\left| \det \left(\mathbf{r}'(u,v,w) \right) \right|}_{\text{absolutbeloppet av Jacobi-determinanten.}} du\, dv\, dw$$

Cylindriska koordinater (r,θ,z).

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

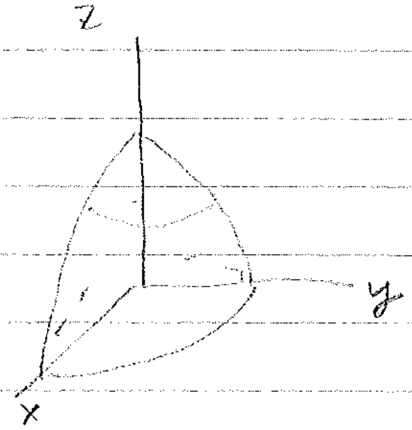
Jacobideterminanten:

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= r(\cos^2\theta + \sin^2\theta) = r$$

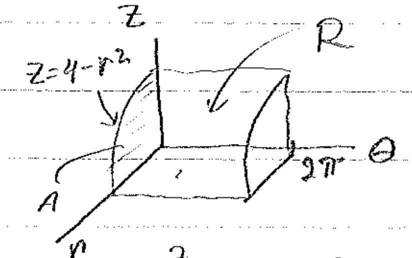
$$dV = r\,dr\,d\theta\,dz$$

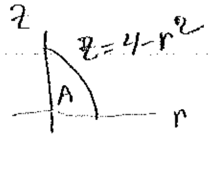
Förra exemplet



$$0 \le z \le 4 - x^2 - y^2 = 4 - r^2$$

Vi får i cyl. koord: $\begin{cases} 0 \le z \le 4-r^2 \\ 0 \le \theta \le 2\pi \end{cases}$





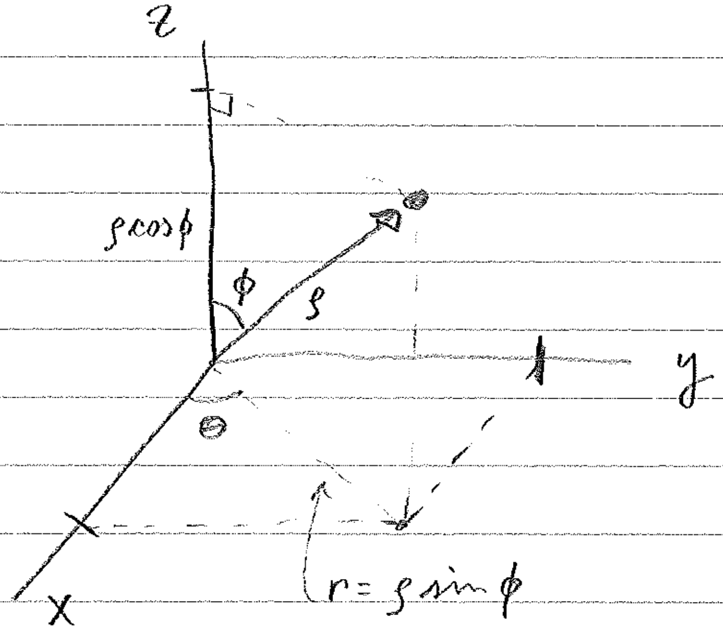
$$I = \iiint_D x\,dx\,dy\,dz = \iiint_R r\cos\theta\; r\,dr\,d\theta\,dz$$

$$= \int_0^{2\pi}\left(\iint_A r\cos\theta \; r\,dr\,dz\right)d\theta$$

$$= \int_0^{2\pi}\cos\theta\,d\theta \int_0^2\left(\int_0^{4-r^2} r^2\,dz\right)dr = 0$$

Sfäriska koordinater (ρ, ϕ, θ).

$$\begin{cases} x = \rho \sin\phi \cos\theta \\ y = \rho \sin\phi \sin\theta \\ z = \rho \cos\phi \end{cases}$$



$$\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} = \begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix} =$$

$$= \dots = \rho^2 \sin\phi$$

obs: $\sin\phi \geq 0$

$$dV = |\rho^2 \sin\phi| \, d\rho \, d\phi \, d\theta = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

<u>Exempel</u> $\iiint\limits_{B(0,1)} (x^2+y^2)\,dx\,dy\,dz = \iiint\limits_{R} (\rho\sin\phi)^2 \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$

$$\begin{pmatrix} 0 \leq \rho \leq 1 \\ 0 \leq \phi \leq \pi \\ 0 \leq \theta \leq 2\pi \end{pmatrix} = \int_0^1 \rho^4 d\rho \int_0^{\pi} \sin^3\phi \, d\phi \int_0^{2\pi} d\theta = \left\{ \begin{array}{l|l} u = -\cos\phi & \phi = 0, u = -1 \\ du = \sin\phi \, d\phi & \phi = \pi, u = 1 \\ \sin^2\phi = 1 - u^2 & \end{array} \right\}$$

$$= \frac{1}{5} \cdot \int_{-1}^{1} (1-u^2)\,du \cdot 2\pi = \frac{1}{5} \cdot \frac{4}{3} \cdot 2\pi = \frac{8\pi}{15}$$