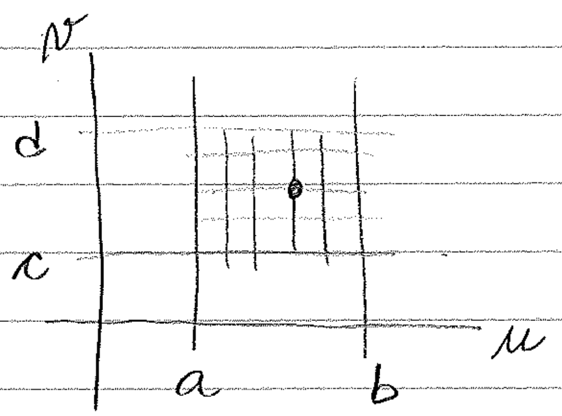
MVE 255 2013-14 Fö 6.3

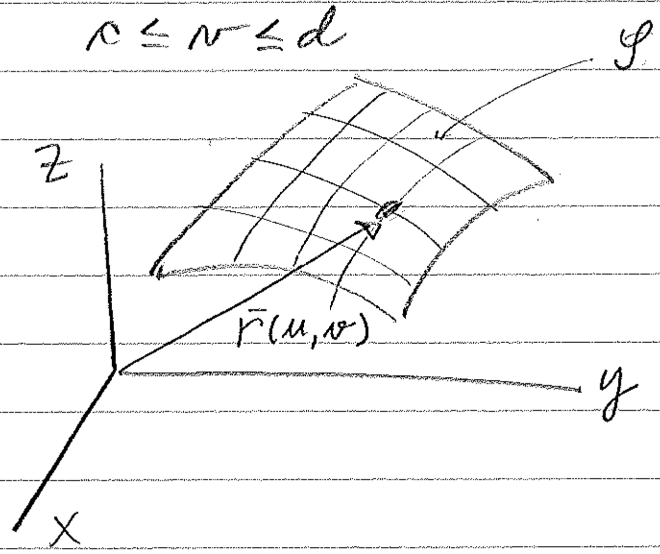
Ytintegral.

Yta S på parameterform:

$\bar{r} = \bar{r}(u,v) \quad a \le u \le b, \; c \le v \le d$

eller $\begin{cases} x = x(u,v) \\ y = y(u,v) \\ z = z(u,v) \end{cases}$ $\quad \begin{matrix} a \le u \le b \\ c \le v \le d \end{matrix}$





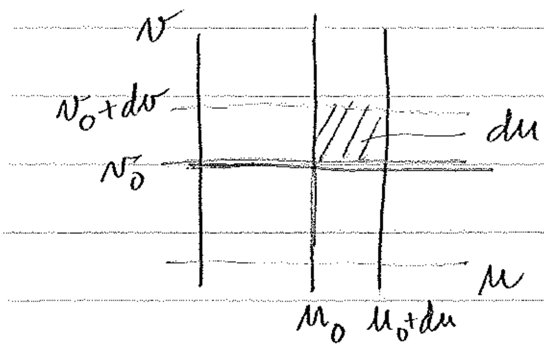
exempel Sfär: $x^2+y^2+z^2=R^2$

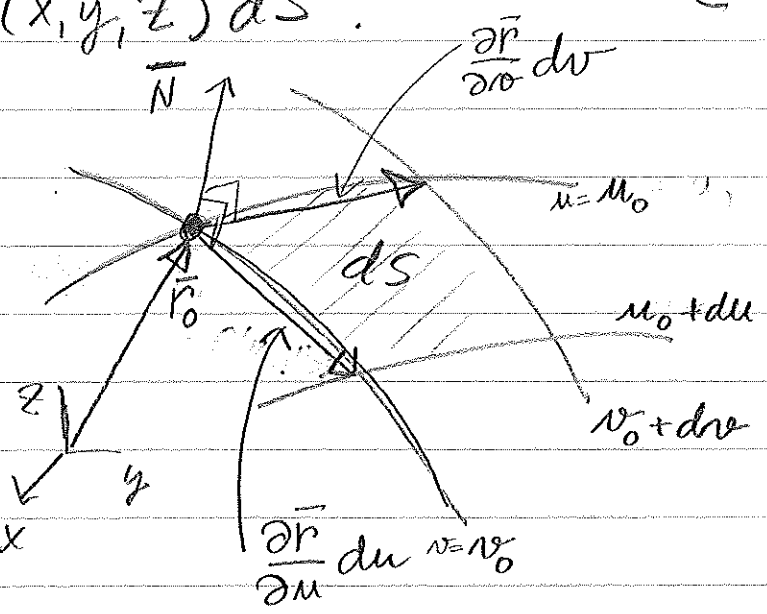
I sfäriska koordinater: $\begin{cases} \rho = R \\ 0 \le \phi \le \pi \\ 0 \le \theta < 2\pi \end{cases}$

Parametrisering:

$$\begin{cases} x = R \sin\phi \cos\theta \\ y = R \sin\phi \sin\theta \\ z = R \cos\phi \end{cases} \quad \begin{matrix} 0 \le \phi \le \pi \\ 0 \le \theta < 2\pi \end{matrix}$$

Ytintegral $\iint_S f(x,y,z)\,dS$.





Areaelementet:

$$dS = \left| \frac{\partial \bar{r}}{\partial u} \times \frac{\partial \bar{r}}{\partial v} \right| du\,dv$$

Obs att $\frac{\partial \bar{r}}{\partial u}(u_0, v_0)$ och $\frac{\partial \bar{r}}{\partial v}(u_0, v_0)$

är tangenter till koordinatkurvorna

$$\bar{r} = \bar{r}(u, v_0), \quad a \le u \le b \quad (v = v_0)$$

$$\bar{r} = \bar{r}(u_0, v), \quad c \le v \le d \quad (u = u_0)$$

De är då också tangenter till ytan S i punkten $\bar{r}_0 = \bar{r}(u_0, v_0)$.

En normalvektor:

$$\bar{N} = \frac{\partial \bar{r}}{\partial u}(u_0, v_0) \times \frac{\partial \bar{r}}{\partial v}(u_0, v_0)$$

Exempel Graf: $z = f(x,y)$

$$\begin{cases} x = u \\ y = v \\ z = f(u,v) \end{cases}$$

Tangenter: $$\begin{cases} \dfrac{\partial \bar{r}}{\partial u} = \bar{i} + 0\bar{j} + f'_x(u,v)\bar{k} \\ \dfrac{\partial \bar{r}}{\partial v} = 0\bar{i} + \bar{j} + f'_y(u,v)\bar{k} \end{cases}$$

En normalvektor

$$\bar{N} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1 & 0 & f'_x \\ 0 & 1 & f'_y \end{vmatrix} = -f'_x\bar{i} - f'_y\bar{j} + \bar{k}$$

$$dS = \left| \frac{\partial \bar{r}}{\partial u} \times \frac{\partial \bar{r}}{\partial v} \right| du\,dv =$$

$$= \sqrt{1 + (f'_x)^2 + (f'_y)^2}\, du\,dv$$

eller $$dS = \sqrt{1 + (f'_x(x,y))^2 + (f'_y(x,y))^2}\, dx\,dy$$

Exempel Sfären.

$$\begin{cases} x = R \sin\phi \cos\theta & 0 \le \phi \le \pi \\ y = R \sin\phi \sin\theta & 0 \le \theta \le 2\pi \\ z = R \cos\phi \end{cases}$$

Tangenter:

$$\begin{cases} \dfrac{\partial \bar{r}}{\partial \phi} = R\cos\phi\cos\theta\,\bar{i} + R\cos\phi\sin\theta\,\bar{j} - R\sin\phi\,\bar{k} \\ \dfrac{\partial \bar{r}}{\partial \theta} = -R\sin\phi\sin\theta\,\bar{i} + R\sin\phi\cos\theta\,\bar{j} + 0\,\bar{k} \end{cases}$$

En normalvektor:

$$\bar{N} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ R\cos\phi\cos\theta & R\cos\phi\sin\theta & -R\sin\phi \\ -R\sin\phi\sin\theta & R\sin\phi\cos\theta & 0 \end{vmatrix} =$$

$$= R^2 \sin^2\phi\cos\theta\,\bar{i} + R^2\sin^2\phi\sin\theta\,\bar{j} +$$

$$+ \left(R^2\sin\phi\cos\phi\cos^2\theta + R^2\sin\phi\cos\phi\sin^2\theta\right)\bar{k}$$

$$= 1$$

$$= R\sin\phi\,\big(R\sin\phi\cos\theta\,\bar{i} + R\sin\phi\sin\theta\,\bar{j} +$$

$$+ R\cos\phi\,\bar{k}\big) \qquad \left(\hat{r} = \frac{\bar{r}}{|\bar{r}|} = \frac{\bar{r}}{R} \text{ enhetsvektor}\right)$$

$$= R\sin\phi\,\bar{r} = R^2\sin\phi\,\frac{\bar{r}}{R} = R^2\sin\phi\,\hat{r}$$

$$dS = |R^2\sin\phi\,\hat{r}|\,d\phi\,d\theta = R^2\sin\phi\,d\phi\,d\theta$$

Kom ihåg $dV = \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$.

Exempel Sfärens area

$$\iint_S dS = \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi} R^2 \sin\phi \, d\phi \, d\theta =$$

$$= R^2 \int_0^{\pi} \sin\phi \, d\phi \int_0^{2\pi} d\theta =$$

$$= R^2 \left[-\cos\phi\right]_0^{\pi} \cdot 2\pi = 4\pi R^2$$

Viktigt: kunna parametrisera och beräkna $\bar{N} = \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}$ och dS i följande fall

- graf $z = f(x,y)$
- cylindriska koordinater
- sfäriska koordinater

Flödesintegral (6.12)

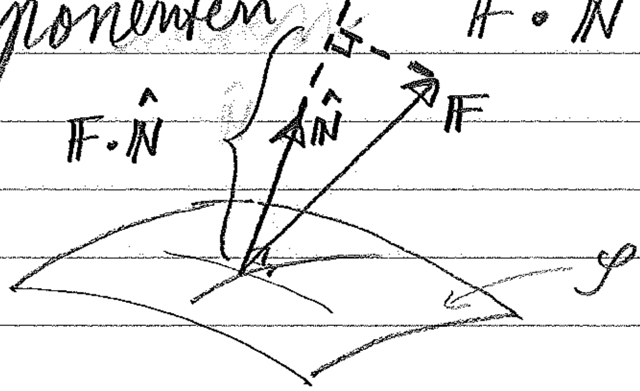
En yta S är <u>orienterbar</u> om det finns en normalvektor $\mathbb{N}(\mathbb{r})$ som varierar kontinuerligt när $\mathbb{r}$ rör sig på ytan.
(Möbius band är ej orienterbart. Om $\mathbb{r}$ går ett varv runt bandet så pekar $\mathbb{N}(\mathbb{r})$ i motsatt riktning när vi kommer tillbaka.)

I så fall väljer vi en <u>orientering</u>, dvs ett enhetsnormalvektorfält $\hat{\mathbb{N}} = \frac{\mathbb{N}}{|\mathbb{N}|}$, och definierar <u>flödesintegralen</u> av vektorfältet $\mathbb{F}$ över S:

$$\iint_S \mathbb{F} \cdot d\mathbb{S} = \iint_S \mathbb{F} \cdot \hat{\mathbb{N}} \, dS .$$

Här är $\hat{\mathbb{N}} = \dfrac{\mathbb{N}}{|\mathbb{N}|}$ enhetsnormalen.

Vi integrerar alltså normalkomponenten $\mathbb{F} \cdot \hat{\mathbb{N}}$ av $\mathbb{F}$:



Jämför tangentkurvintegralen:

$$\int_C \mathbb{F} \cdot d\mathbb{r} = \int_C \mathbb{F} \cdot \hat{\mathbb{T}}\, ds$$

<u>Exempel</u> Hastighetsfält $\mathbb{v}$.

Volymsflöde genom $\mathcal{S}$:

$$\iint_{\mathcal{S}} \underbrace{\mathbb{v}}_{\left[\frac{m}{s}\right]} \cdot \hat{\mathbb{N}} \underbrace{dS}_{\left[m^2\right]} \qquad \left[\frac{m^3}{s}\right]$$

Massflöde: $\displaystyle\iint_{\mathcal{S}} \underset{\substack{\uparrow \\ \text{densitet}\left[\frac{kg}{m^3}\right]}}{\delta}\, \mathbb{v} \cdot \hat{\mathbb{N}}\, dS \qquad \left[\frac{kg}{s}\right]$

Ytintegral och flödesintegral

$S: \quad \bar{r} = \bar{r}(u,v), \quad (u,v) \in D$

- tangenter: $\frac{\partial \bar{r}}{\partial u}, \frac{\partial \bar{r}}{\partial v}$
- normalvektor: $\bar{N} = \pm \frac{\partial \bar{r}}{\partial u} \times \frac{\partial \bar{r}}{\partial v}$ (in/ut, upp/ner)
- enhetsnormal: $\hat{N} = \frac{\bar{N}}{|\bar{N}|}$
- ytelement: $dS = \left| \frac{\partial \bar{r}}{\partial u} \times \frac{\partial \bar{r}}{\partial v} \right| du\,dv = |\bar{N}|\, du\,dv$
- normalytelement:

$$d\bar{S} = \pm \frac{\partial \bar{r}}{\partial u} \times \frac{\partial \bar{r}}{\partial v}\, du\,dv = \bar{N}\, du\,dv = \hat{N}\, dS$$

välj orientering med + eller −

Exempel Sfär: $x^2 + y^2 + z^2 = R^2$

Sfäriska koordinater: $\rho = R$

$$\begin{cases} x = R \sin\phi \cos\theta & \theta \in [0, 2\pi] \\ y = R \sin\phi \sin\theta & \phi \in [0, \pi] \\ z = R \cos\phi & \end{cases}$$

$$\frac{\partial \bar{r}}{\partial \phi} = R\cos\phi\cos\theta\, \bar{i} + R\cos\phi\sin\theta\, \bar{j} - R\sin\phi\, \bar{k}$$

$$\frac{\partial \bar{r}}{\partial \theta} = -R\sin\phi\sin\theta\, \bar{i} + R\sin\phi\cos\theta\, \bar{j}$$

$$\frac{\partial \bar{r}}{\partial \phi} \times \frac{\partial \bar{r}}{\partial \theta} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ R\cos\phi\cos\theta & R\cos\phi\sin\theta & -R\sin\phi \\ -R\sin\phi\sin\theta & R\sin\phi\cos\theta & 0 \end{vmatrix}$$

$$= R^2\sin\phi \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \cos\phi\cos\theta & \cos\phi\sin\theta & -\sin\phi \\ -\sin\theta & \cos\theta & 0 \end{vmatrix}$$

$$= R^2\sin\phi \underbrace{\left(\sin\phi\cos\theta\,\bar{i} + \sin\phi\sin\theta\,\bar{j} + \cos\phi\,\bar{k} \right)}_{= \frac{\bar{r}}{R} = \frac{\bar{r}}{|\bar{r}|} = \hat{\bar{r}}}$$

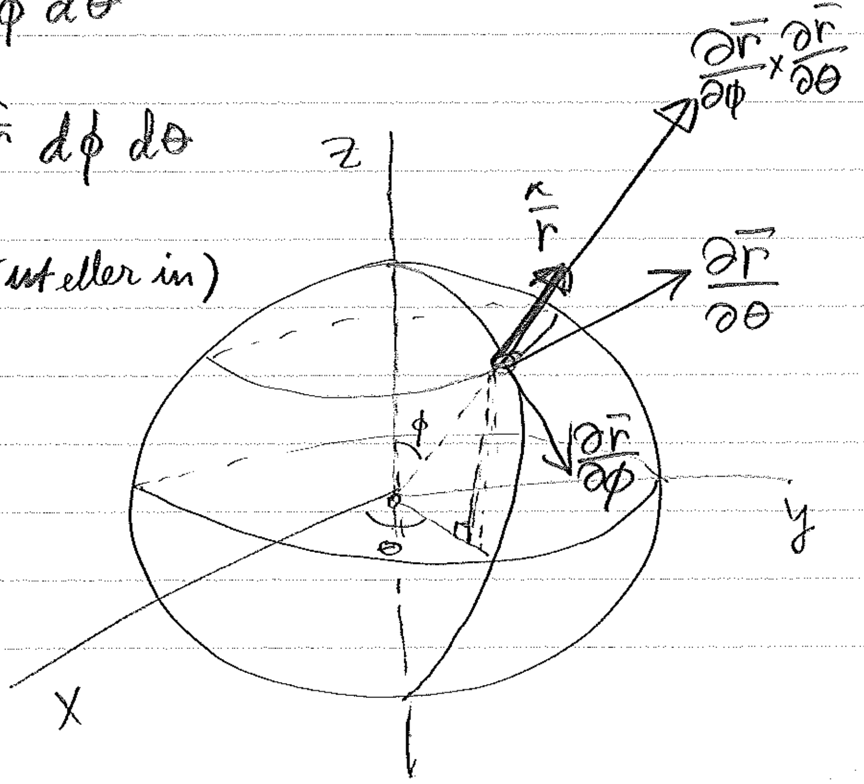
$$= R^2\sin\phi\ \hat{\bar{r}}$$

$$\left|\frac{\partial \bar{r}}{\partial \phi} \times \frac{\partial \bar{r}}{\partial \theta}\right| = R^2\sin\phi \qquad \left(\text{obs: } \sin\phi \geq 0 \text{ då } 0 \leq \phi \leq \pi\right)$$

$$dS = R^2\sin\phi\, d\phi\, d\theta$$

$$d\bar{S} = \pm R^2\sin\phi\ \hat{\bar{r}}\, d\phi\, d\theta$$

↑ välj orientering (ut eller in)



Grad, div, rot. (6.13)

Med nabla-operatorn

$$\nabla = \frac{\partial}{\partial x}\mathbb{i} + \frac{\partial}{\partial y}\mathbb{j} + \frac{\partial}{\partial z}\mathbb{k}$$

kan vi bilda

gradienten av skalärt fält ϕ:

$$\nabla\phi = \frac{\partial\phi}{\partial x}\mathbb{i} + \frac{\partial\phi}{\partial y}\mathbb{j} + \frac{\partial\phi}{\partial z}\mathbb{k}$$

och divergensen och rotationen

av vektorfält $\mathbb{F}$:

$$\nabla\cdot\mathbb{F} = \operatorname{div}\mathbb{F} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\cdot(F_1, F_2, F_3)$$

$$= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

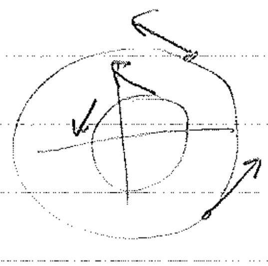
$$\nabla\times\mathbb{F} = \operatorname{rot}\mathbb{F} = \operatorname{curl}\mathbb{F}$$

$$= \begin{vmatrix} \mathbb{i} & \mathbb{j} & \mathbb{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

<u>Exempel</u> Stelkroppsrotation

$$\bar{v} = \Omega\left(-y\,\bar{i} + x\,\bar{j}\right)$$



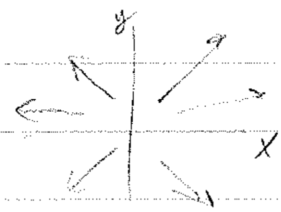
$$\bar{\nabla}\cdot\bar{v} = \Omega\left(\frac{\partial}{\partial x}(-y) + \frac{\partial}{\partial y}x\right) = 0$$

$$\bar{\nabla}\times\bar{v} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\Omega y & \Omega x & 0 \end{vmatrix} = 0\bar{i} + 0\bar{j} + 2\Omega\bar{k}$$

$$= 2\Omega\bar{k}$$

Divergerar inte, men roterar.

<u>Exempel</u> $\bar{F} = x\,\bar{i} + y\,\bar{j} + z\,\bar{k}$



$$\bar{\nabla}\cdot\bar{F} = 1 + 1 + 1 = 3$$

$$\bar{\nabla}\times\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = 0\bar{i} + 0\bar{j} + 0\bar{k}$$

Divergerar, Roterar ej.

Några räkneregler

1) $\nabla(\phi \Psi) = \Psi \nabla\phi + \phi \nabla\Psi$

2) $\nabla \cdot (\phi \mathbb{F}) = \nabla\phi \cdot \mathbb{F} + \phi \nabla \cdot \mathbb{F}$

3) $\nabla \times (\phi \mathbb{F}) = \nabla\phi \times \mathbb{F} + \phi \nabla \times \mathbb{F}$

4) $\nabla \cdot (\nabla \times \mathbb{F}) = 0$ div rot $\mathbb{F} = 0$

5) $\nabla \times \nabla\phi = \mathbb{0}$ rot grad $\phi = \mathbb{0}$

Bevisa dessa! (Skriv på komponentform och räkna på...)

Exempel: 2) $\nabla \cdot (\phi \mathbb{F}) = \nabla \cdot (\phi F_1, \phi F_2, \phi F_3) =$

$$= \frac{\partial}{\partial x}(\phi F_1) + \frac{\partial}{\partial y}(\phi F_2) + \frac{\partial}{\partial z}(\phi F_3) =$$

$$= \frac{\partial \phi}{\partial x} F_1 + \phi \frac{\partial F_1}{\partial x} + \frac{\partial \phi}{\partial y} F_2 + \phi \frac{\partial F_2}{\partial y} + \frac{\partial \phi}{\partial z} F_3 + \phi \frac{\partial F_3}{\partial z}$$

$$= \frac{\partial \phi}{\partial x} F_1 + \frac{\partial \phi}{\partial y} F_2 + \frac{\partial \phi}{\partial z} F_3$$

$$+ \phi \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) =$$

$$= \nabla\phi \cdot \mathbb{F} + \phi \nabla \cdot \mathbb{F}$$