MVE 255 FÖ 5.1

Idag: Variabelbyte 14.4
Trippelintegralen 14.5

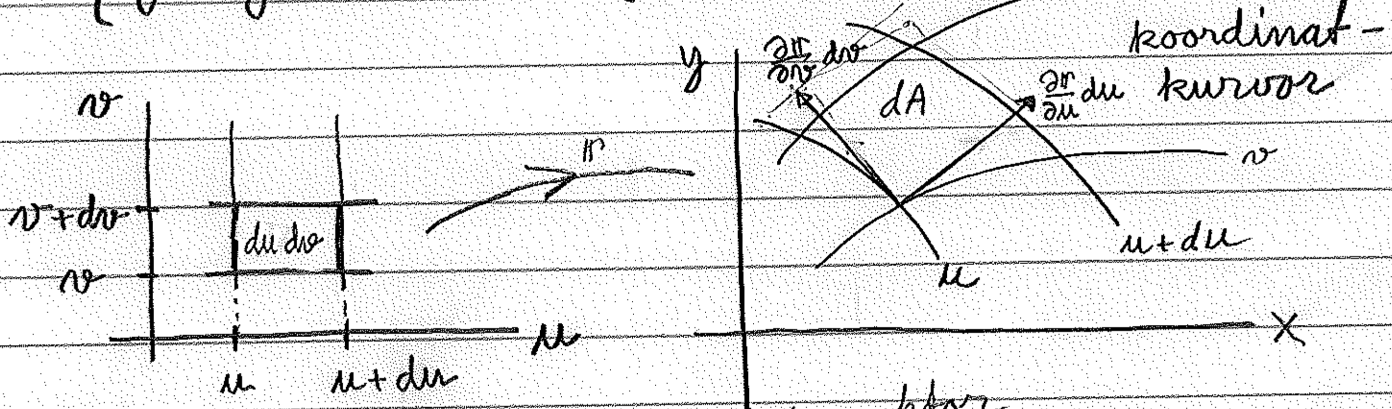
## Variabelbyte

Nya koordinater $(u, v)$.

$\mathbf{r} = \mathbf{r}(u, v)$

Transformation: $\begin{cases} x = x(u,v) \\ y = y(u,v) \end{cases}$

Invers transformation: $\begin{cases} u = u(x,y) \\ v = v(x,y) \end{cases}$




$$dA = \left| \frac{\partial \mathbf{r}}{\partial u} du \times \frac{\partial \mathbf{r}}{\partial v} dv \right|$$ längd av vektor

areaelementet

spänns upp av tangenterna $\frac{\partial \mathbf{r}}{\partial u} du$ och $\frac{\partial \mathbf{r}}{\partial v} dv$.

$$\frac{\partial \mathbf{r}}{\partial u} du \times \frac{\partial \mathbf{r}}{\partial v} dv = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial x}{\partial u} du & \frac{\partial y}{\partial u} du & 0 \\ \frac{\partial x}{\partial v} dv & \frac{\partial y}{\partial v} dv & 0 \end{vmatrix} = \left\{ \begin{matrix} \text{bryt ut} \\ du \text{ och } dv \end{matrix} \right\}$$

$$= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} du\, dv\, \mathbf{k} = \frac{\partial(x,y)}{\partial(u,v)} du\, dv\, \mathbf{k}$$

Jacobideterminanten för transformationen:

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\left(\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{bmatrix}\right) = \underbrace{\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}}_{=\text{ transponatet } \mathbf{r}'(u,v)^T \text{ av Jacobimatrisen}} \overset{(\det(A^T)=\det(A))}{=}$$

$$= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \det(\mathbf{r}'(u,v))$$

determinant-streck

$= $ Jacobimatrisen $\mathbf{r}'(u,v)$

Vi får nu

$$dA = \left|\frac{\partial \mathbf{r}}{\partial u}\,du \times \frac{\partial \mathbf{r}}{\partial v}\,dv\right| = \left|\frac{\partial(x,y)}{\partial(u,v)}\,du\,dv\,\mathbf{k}\right| =$$

längd

abs. belopp

$$= \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

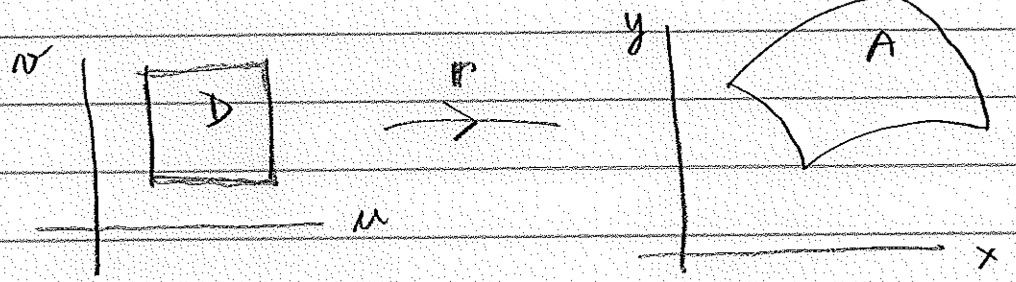
dvs

$$\boxed{\begin{aligned} dx\,dy &= \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv \\ &= |\det(\mathbf{r}'(u,v))|\,du\,dv \end{aligned}}$$

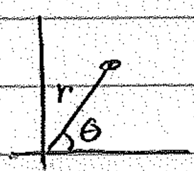
Obs: $\left|\frac{\partial(x,y)}{\partial(u,v)}\right|$ är yt-skalan vid transformationen.

**Sats** (Variabelbyte i dubbelintegral)

$$\iint_A f(x,y)\,dx\,dy = \iint_D f(\mathbf{r}(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$




**Exempel** Polära koord $(r,\theta)$.




$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \qquad \begin{cases} r = \sqrt{x^2+y^2} \\ \theta = \arctan\frac{y}{x} \end{cases}$$

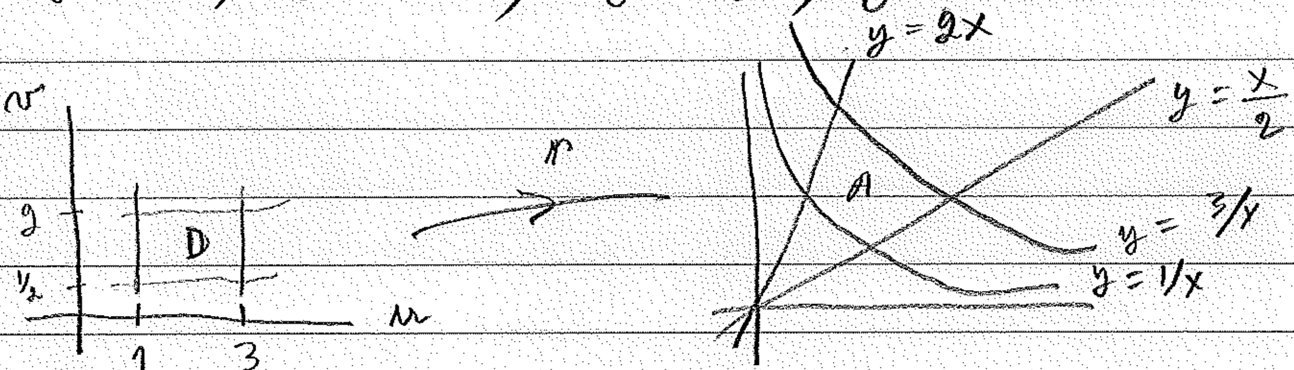
$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = r$$

$$dA = \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| dr\,d\theta = |r|\,dr\,d\theta = r\,dr\,d\theta.$$

## Exempel

Beräkna $\iint_A \frac{x}{y}\,dx\,dy$ där A är området mellan kurvorna

$$y = \frac{1}{x},\quad y = \frac{3}{x},\quad y = \frac{x}{2},\quad y = 2x.$$




Vi har $1 \le xy \le 3$, $\frac{1}{2} \le \frac{y}{x} \le 2$. Detta bestämmer en rektangel D i uv-planet.
Välj därför $u = xy$, $v = y/x$. Detta är inversa transformationen. Lös ut x, y.

Ekv (2): $y = vx$. I ekv (1): $u = vx^2$, $x = (\pm)\sqrt{\frac{u}{v}}$
Sedan $y = v\sqrt{\frac{u}{v}} = \sqrt{uv}$.
Transformationen blir

$$\begin{cases} x = \sqrt{\frac{u}{v}} \\ y = \sqrt{uv} \end{cases}$$

Jacobideterminanten: $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} =$

$$= \begin{vmatrix} \frac{1}{2\sqrt{uv}} & -\frac{\sqrt{u}}{2\sqrt{v^3}} \\ \frac{\sqrt{v}}{2\sqrt{u}} & \frac{\sqrt{u}}{2\sqrt{v}} \end{vmatrix} = \frac{1}{2v}$$

Integralen blir

$$\iint_A \frac{x}{y}\,dx\,dy = \iint_D \frac{\sqrt{\frac{u}{v}}}{\sqrt{uv}} \frac{1}{2v}\,du\,dv =$$

$$= \iint_D \frac{1}{2v^2}\,du\,dv = \int_1^3 \left( \int_{1/2}^2 \frac{1}{2v^2}\,dv \right) du$$

$$= \int_1^3 du \int_{1/2}^2 \frac{1}{2v^2}\,dv = 2 \cdot \left[ \frac{1}{-2v} \right]_{1/2}^2$$

$$= 2\left( \frac{1}{-4} - \frac{1}{-1} \right) = 2 \cdot \frac{3}{4} = \frac{3}{2}$$

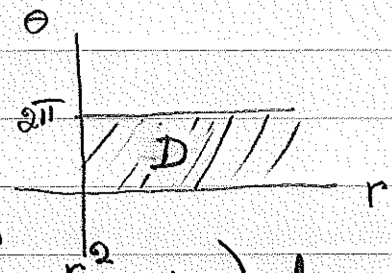
<u>Exempel</u> $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$

<u>Bevis</u> Låt $I = \int_{-\infty}^{\infty} e^{-x^2}\,dx$. Då blir

$$I^2 = \int_{-\infty}^{\infty} e^{-x^2}\,dx \int_{-\infty}^{\infty} e^{-y^2}\,dy =$$

$$= \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} e^{-x^2} e^{-y^2}\,dx \right) dy$$

$$= \iint_{\mathbb{R}^2} e^{-x^2-y^2}\,dx\,dy$$




$$\overset{\text{polära}}{=} \iint_{D} e^{-r^2}\, r\,dr\,d\theta = \int_0^{2\pi} \left( \int_0^{\infty} e^{-r^2}\, r\,dr \right) d\theta$$

(generaliserad integral
positiv integrand
upprepad integration)

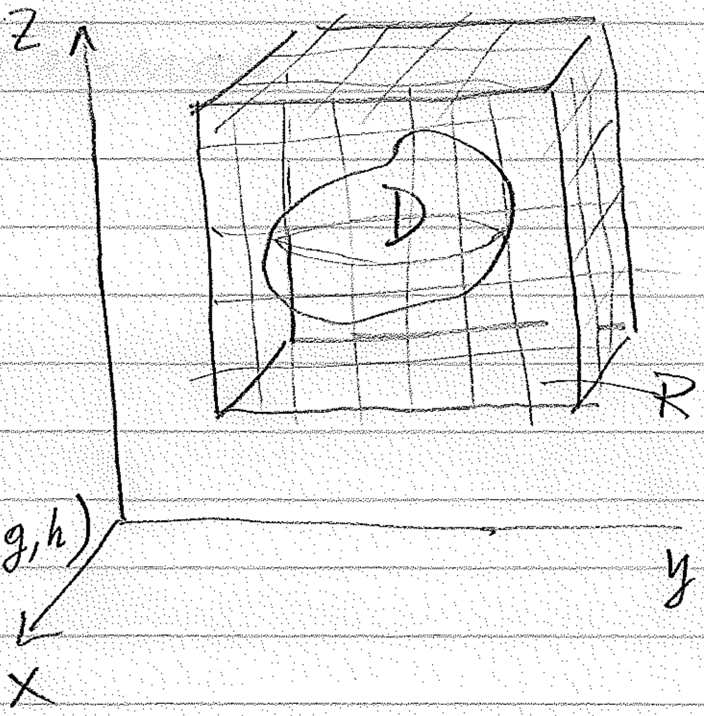
$$= \int_0^{2\pi} d\theta \int_0^{\infty} e^{-r^2}\, r\,dr$$

$$= 2\pi \left[ -\tfrac{1}{2} e^{-r^2} \right]_0^{\infty} = 2\pi \cdot \tfrac{1}{2} = \pi$$

# Trippelintegralen (14.5)

$$\iiint_D f \, dV = \iiint_D f(x,y,z)\, dx\, dy\, dz$$

definieras med partition av området och Riemann-summa.



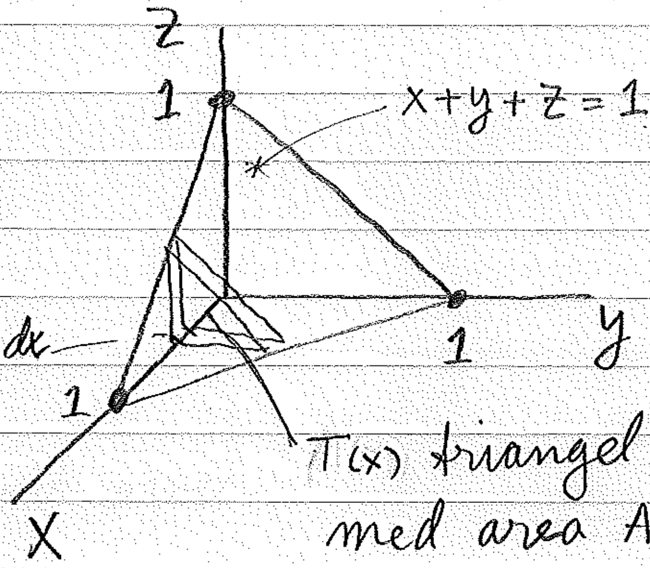

Först över råtblock R, sedan allmänt område D

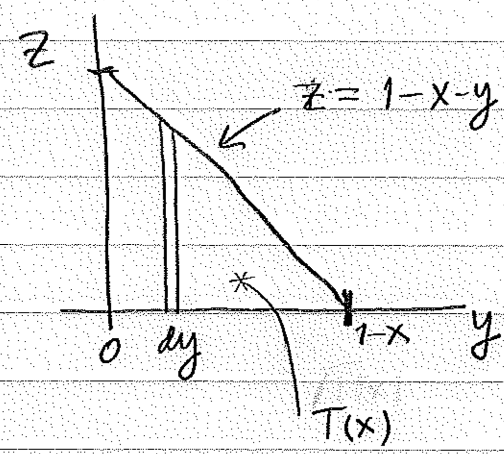
Matlab:

integral3(f, a, b, c, d, g, h)
integrerar över råtblock.

Kan ibland beräknas med upprepad integration.

**Exempel** Volymen av tetraeder.




Tetraedern D är enkel i alla riktningar.

$$V = \iiint_T 1\, dx\,dy\,dz = \int_0^1 A(x)\, dx$$

$$= \int_0^1 \left( \iint_{T(x)} 1\, dy\,dz \right) = \int_0^1 \left( \int_0^{1-x} \left( \int_0^{1-x-y} 1\, dz \right) dy \right) dx$$

$$= \int_0^1 \left( \int_0^{1-x} \left[ z \right]_0^{1-x-y} dy \right) dx$$

$$= \int_0^1 \left( \int_0^{1-x} (1-x-y)\, dy \right) dx = \int_0^1 \left[ (1-x)y - \tfrac{1}{2}y^2 \right]_0^{1-x} dx$$

$$= \tfrac{1}{2} \int_0^1 (1-x)^2\, dx = \tfrac{1}{2} \left[ \frac{-(1-x)^3}{3} \right]_0^1 = \frac{1}{6}.$$