

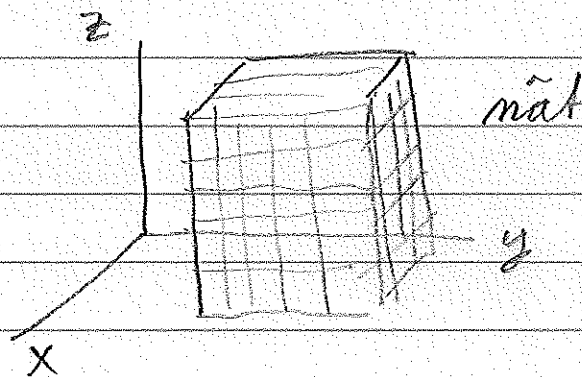
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Trippelintegraler 14.5

$$\iiint_R f(x, y, z) dx dy dz = \lim_{I \rightarrow 0} \sum_{i=1}^I \sum_{j=1}^J \sum_{k=1}^K f(x_{ijk}) \Delta x_i \Delta y_j \Delta z_k$$

R rektangulärt område i $\mathbb{R}^3$ (rätblock).



Allmänt begränsat område D :

integrera över stort rekt. område R så att $D \subset R$, sätt f till 0 utanför D .

Kan ibland räknas ut med upprepad integration.

Rektangulärt: $R = [a_1, a_2] \times [b_1, b_2] \times [c_1, c_2]$

$$\iiint_R f(x, y, z) dx dy dz = \int_{a_1}^{a_2} \left(\int_{b_1}^{b_2} \left(\int_{c_1}^{c_2} f(x, y, z) dz \right) dy \right) dx$$

rektangulärt, upprepad integration i vilken ordning som helst.

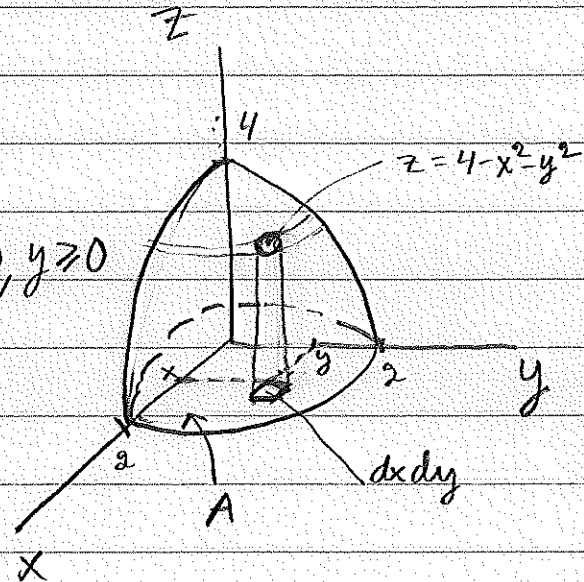
Altmanant område är oftast svårt.

Exempel

$$I = \iiint_D x \, dx \, dy \, dz$$

$$D: 0 \leq z \leq 4 - x^2 - y^2, \quad x \geq 0, y \geq 0$$

Enkelt i z , dvs
mellan två grafer
 $z=0$ och $z=4-x^2-y^2$.



$$I = \iint_A \left(\int_0^{4-x^2-y^2} x \, dz \right) dx \, dy$$

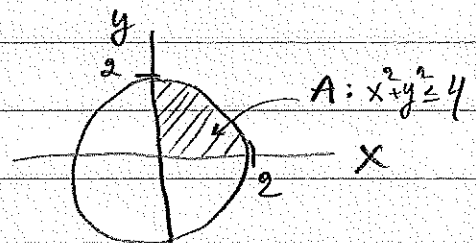
$$= \iint_A \left(x \left[z \right]_0^{4-x^2-y^2} \right) dx \, dy$$

$$= \iint_A x(4-x^2-y^2) \, dx \, dy = \{ \text{polära} \} =$$

$$= \int_0^{\pi/2} \int_0^2 r \cos \theta (4-r^2) r \, dr \, d\theta$$

$$= \int_0^{\pi/2} \cos \theta \, d\theta \int_0^2 r^2(4-r^2) \, dr = \underbrace{[\sin \theta]_0^{\pi/2}}_{=1} \underbrace{\int_0^2 (4r^2 - r^4) \, dr}_{= \frac{64}{15}} = \frac{64}{15}$$

Bottenytan A:



J Matlab :

$$\gg f = @(x,y,z) (x.* (z \leq 4 - x.^2 - y.^2))$$

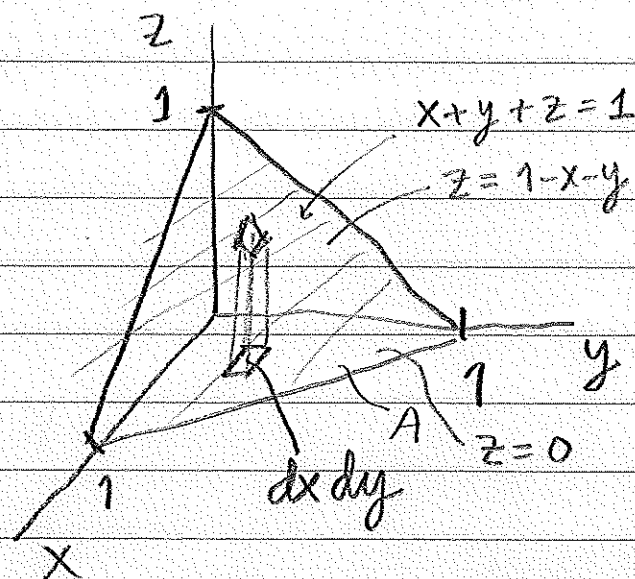
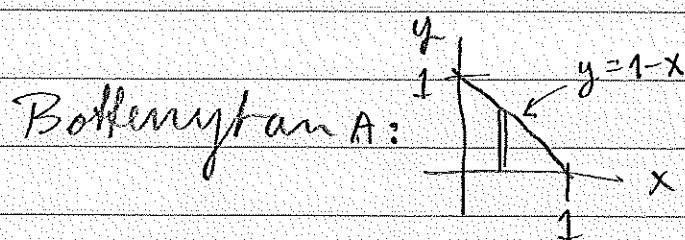
$$\gg I = \text{integral3}(f, \underbrace{0, 2, 0, 2, 0, 4})$$

$$[0, 2] \times [0, 2] \times [0, 4]$$

Se mer detaljer på sista sidan.

Exempel Volymen av tetraeder.

T begränsas av
planet $x+y+z=1$ och
av koord. planen
 $x=0$, $y=0$ och $z=0$.



Enkelt i z : $0 \leq z \leq 1-x-y$ (mellan 2 grafer)

$$V = \iiint_T 1 \, dx \, dy \, dz = \iint_A \left(\int_0^{1-x-y} 1 \, dz \right) dx \, dy$$

$$= \int_0^1 \left(\int_0^{1-x} \left(\int_0^{1-x-y} dz \right) dy \right) dx$$

$$= \int_0^1 \int_0^{1-x} (1-x-y) \, dy \, dx$$

$$= \int_0^1 \left[(1-x)y - \frac{1}{2}y^2 \right]_0^{1-x} dx$$

$$= \int_0^1 \left((1-x)^2 - \frac{1}{2}(1-x)^2 \right) dx$$

$$= \frac{1}{2} \int_0^1 (1-x)^2 dx = \frac{1}{2} \left[-\frac{1}{3}(1-x)^3 \right]_0^1 = \frac{1}{6}.$$

I FÖ 5.1 beräknas denna med "skivmetoden".

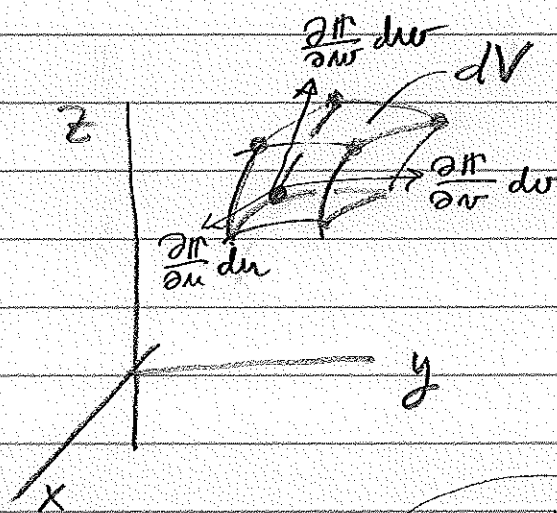
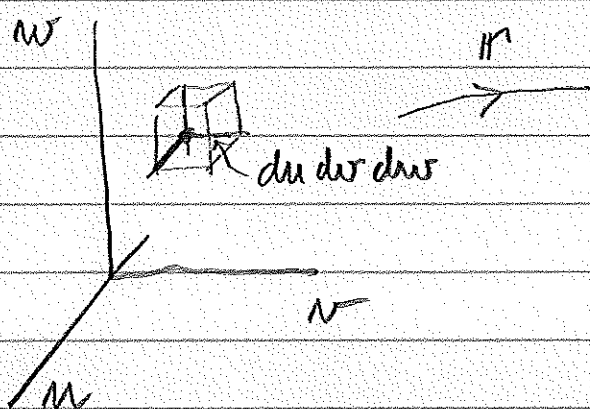
6.10 Variabelbytte. 14.6

(4

Nya koordinater (u, v, w) .

$$\begin{cases} x = x(u, v, w) \\ y = y(u, v, w) \\ z = z(u, v, w) \end{cases}$$

$$r = r(u, v, w)$$



dV spänns upp av tangenterna.

$$dV = \left| \frac{\partial r}{\partial u} du \cdot \left(\frac{\partial r}{\partial v} dv \times \frac{\partial r}{\partial w} dw \right) \right| = \text{absolutbelopp av trippelprodukten}$$

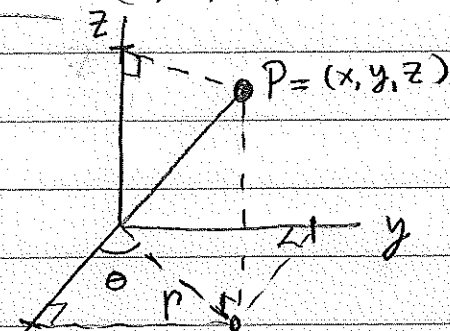
$$= \left| \frac{\partial (x, y, z)}{\partial (u, v, w)} \right| du dv dw =$$

$$= \left| \det \left(r'(u, v, w) \right) \right| du dv dw$$

absolutbeloppet av Jacobi-determinanten.

Cylindriska koordinater (r, θ, z) .

$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \\ z = z \end{cases}$$



Jacobi determinanter:

$$\frac{\partial(x, y, z)}{\partial(r, \theta, z)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos \theta & -r \sin \theta & 0 \\ \sin \theta & r \cos \theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

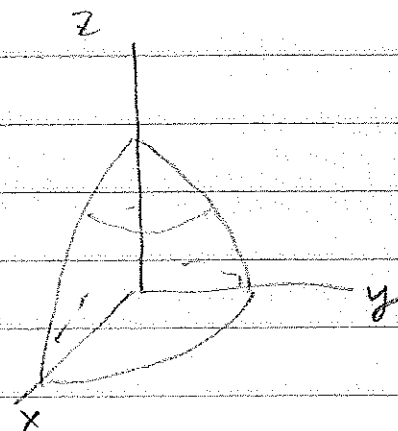
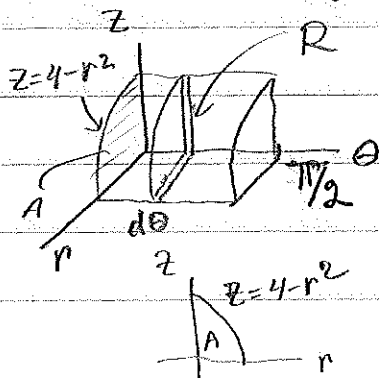
$$= r(\cos^2 \theta + \sin^2 \theta) = r$$

$$dV = r dr d\theta dz$$

Första exemplet

$$0 \leq z \leq 4 - x^2 - y^2 = 4 - r^2$$

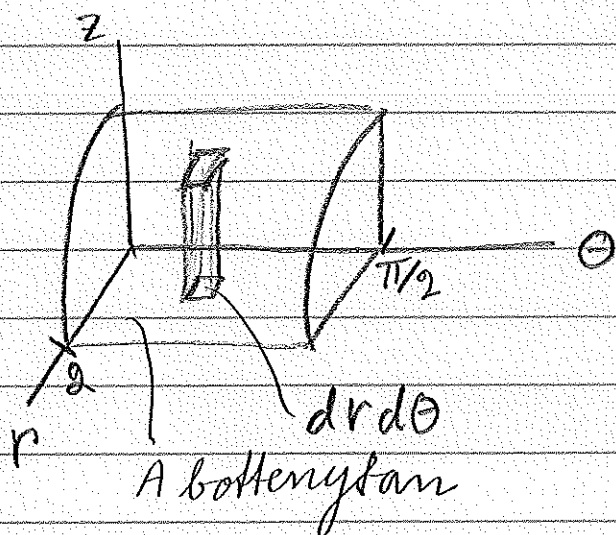
Vi får i cyl. koord: $\begin{cases} 0 \leq z \leq 4 - r^2 \\ 0 \leq \theta \leq \pi/2 \end{cases}$



$$\begin{aligned} I &= \iiint_D x \, dx \, dy \, dz = \iiint_R r \cos \theta \, r \, dr \, d\theta \, dz \\ &= \int_0^{\pi/2} \left(\iint_A r \cos \theta \, r \, dr \, dz \right) d\theta \\ &= \int_0^{\pi/2} \cos \theta \, d\theta \int_0^2 \left(\int_0^{4-r^2} r^2 \, dz \right) dr = 1 \cdot \int_0^2 r^2 (4 - r^2) dr = 64/15 \end{aligned}$$

Detta var "skivmetoden".

Alternativ: R är enkelt i z



$$0 \leq z \leq 4 - r^2, (r, \theta) \in A$$

$$I = \iiint_R r \cos \theta \, r \, dr \, d\theta \, dz = \iint_A \left(\int_0^{4-r^2} r^2 \cos \theta \, dz \right) dr \, d\theta$$

$$= \iint_A r^2 \cos \theta \left[z \right]_0^{4-r^2} dr \, d\theta$$

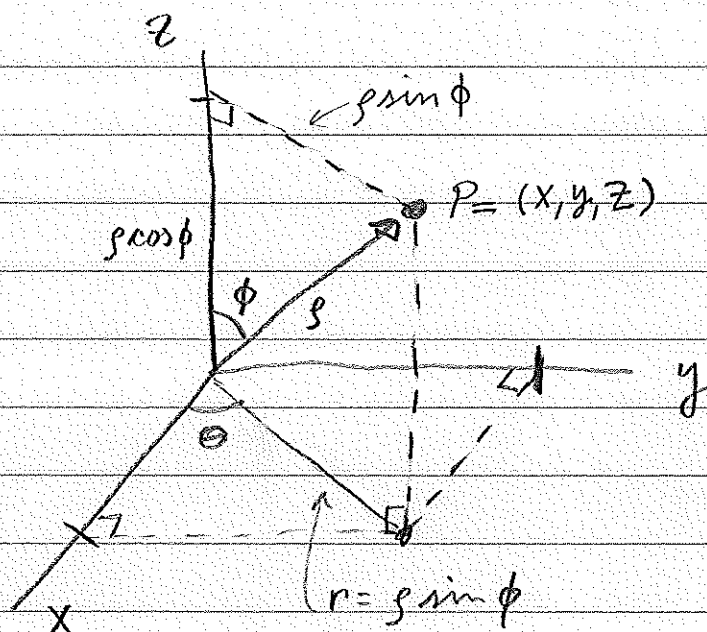
$$= \iint_A (4 - r^2) r^2 \cos \theta \, dr \, d\theta = \left\{ A \text{ är rektangel} \right\}$$

$$= \int_0^{\pi/2} \left(\int_0^2 (4 - r^2) r^2 \cos \theta \, dr \right) d\theta$$

$$= \int_0^{\pi/2} \cos \theta \, d\theta \int_0^2 (4r^2 - r^4) \, dr = 1 \cdot \frac{64}{15} = \frac{64}{15}$$

Sfäriska koordinater (ρ, ϕ, θ) .

$$\begin{cases} x = \rho \sin \phi \cos \theta \\ y = \rho \sin \phi \sin \theta \\ z = \rho \cos \phi \end{cases}$$



$$\frac{\partial(x, y, z)}{\partial(\rho, \phi, \theta)} = \begin{vmatrix} \sin \phi \cos \theta & \rho \cos \phi \cos \theta & -\rho \sin \phi \sin \theta \\ \sin \phi \sin \theta & \rho \cos \phi \sin \theta & \rho \sin \phi \cos \theta \\ \cos \phi & -\rho \sin \phi & 0 \end{vmatrix} =$$

$$= \dots = \rho^2 \sin \phi$$

obs: $\sin \phi \geq 0$

$$dV = |\rho^2 \sin \phi| d\rho d\phi d\theta = \rho^2 \sin \phi d\rho d\phi d\theta$$

Exempel $\iiint_{B(0;1)} (x^2 + y^2) dx dy dz = \iiint_{\mathcal{R}} (\rho \sin \phi)^2 \rho^2 \sin \phi d\rho d\phi d\theta$

$$\begin{pmatrix} 0 \leq \rho \leq 1 \\ 0 \leq \phi \leq \pi \\ 0 \leq \theta \leq 2\pi \end{pmatrix} = \int_0^1 \rho^4 d\rho \int_0^\pi \sin^3 \phi d\phi \int_0^{2\pi} d\theta = \begin{cases} u = -\cos \phi \\ du = \sin \phi d\phi \\ \sin^2 \phi = 1 - u^2 \end{cases} \begin{matrix} \phi=0, u=-1 \\ \phi=\pi, u=1 \end{matrix}$$

$$= \frac{1}{5} \cdot \int_{-1}^1 (1 - u^2) du \cdot 2\pi = \frac{1}{5} \cdot \frac{4}{3} \cdot 2\pi = \frac{8\pi}{15}$$

14.7 Endast "Moments and Centres of Mass." sid 850-854

Masscentrum: $(\bar{x}, \bar{y}, \bar{z})$ där

$$\bar{x} = \frac{\iiint_R x \delta(x, y, z) dV}{\iiint_R \delta(x, y, z) dV}$$

δ = massfäthet =
= densitet
[kg/m³]

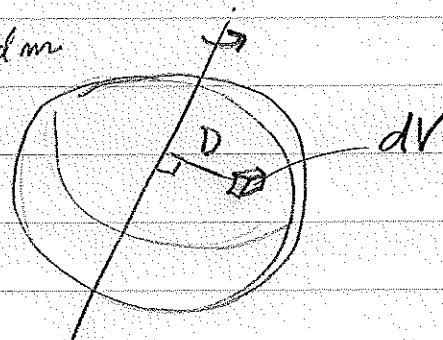
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På vektorform: $\bar{r} = \frac{\iiint_R r \delta dV}{\iiint_R \delta dV} = \frac{\iiint_R r dm}{\iiint_R dm}$

Tröghetsmoment m.a.p. axel:

$$I = \iiint_R D^2 \delta(x, y, z) dV = \iiint_R D^2 dm$$

D = avståndet till axeln



$$dV = r^2 \sin \phi \, dr \, d\phi \, d\theta \quad (\text{obs: } \sin \phi \geq 0)$$

Exempel. Tröghetsmomentet för enhetsklotet m. a. p. z-axeln: $\left(\begin{array}{l} \delta = \text{massfäthet} \\ \left[\frac{\text{kg}}{\text{m}^3} \right] \end{array} \right)$

$$I = \iiint_B (x^2 + y^2) \delta \, dV = \delta \iiint_0^{2\pi} \iiint_0^{\pi} (r^2 \sin^2 \phi) r^2 \sin \phi \, dr \, d\phi \, d\theta$$

$$= \delta \int_0^1 r^4 \, dr \int_0^{\pi} \sin^3 \phi \, d\phi \int_0^{2\pi} d\theta = \left\{ \begin{array}{l} u = -\cos \phi \\ du = +\sin \phi \, d\phi \\ \sin^2 \phi = 1 - u^2 \\ \phi = 0 \Rightarrow u = -1, \phi = \pi \Rightarrow u = 1 \end{array} \right\}$$

$$= \delta \frac{1}{5} \cdot \int_{-1}^1 (1 - u^2) \, du \cdot 2\pi = \delta \frac{1}{5} \cdot \frac{4}{3} \cdot 2\pi = \frac{8\pi}{15} \delta$$

$$\delta = \text{massfäthet} = \text{konstant} \left[\frac{\text{kg}}{\text{m}^3} \right]$$

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Example lecture 5.2. Repeated integration with integral3.

```
format long
xmin = 0;
xmax = 2;
ymin = 0;
ymax = @(x) sqrt(4 - x.^2);
zmin = 0;
zmax = @(x,y) 4 - x.^2 - y.^2;

f=@(x,y,z) x;
I=integral3(f,xmin,xmax,ymin,ymax,zmin,zmax)
Iexact=64/15
```

I =

4.266666666666317

Iexact =

4.266666666666667

Simpler alternative with triplequad (much less accurate)

```
ff=@(x,y,z) (x.*( z<= 4-x.^2-y.^2) );
II=triplequad(ff,0,2,0,2,0,4)
```

II =

4.266558675930201

Integral3 cannot handle this for some reason.

```
ff=@(x,y,z) (x.*( z<= 4-x.^2-y.^2) );
III=integral3(ff,0,2,0,2,0,4)
```

Warning: Reached the maximum number of function evaluations (10000). The result fails the global error test.

Warning: The integration was unsuccessful.

III =

NaN

.....