

Kurwintegraler

$$C: \mathbf{r} = \mathbf{r}(t), \quad t \in [a, b]$$

Tangent: $\mathbf{T} = \mathbf{r}'(t)$, Einheits tangent: $\hat{\mathbf{T}} = \frac{\mathbf{T}}{|\mathbf{T}|}$

$$ds = |\mathbf{r}'(t)| dt$$

$$d\mathbf{r} = \hat{\mathbf{T}} ds = \mathbf{r}'(t) dt$$

$$\begin{aligned} \int_C f ds &= \int_a^b f(\mathbf{r}(t)) |\mathbf{r}'(t)| dt; \quad \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \hat{\mathbf{T}} ds = \\ &= \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt \end{aligned}$$

Flächintegrals

$$\mathcal{S}: \mathbf{r} = \mathbf{r}(u, v), \quad (u, v) \in D$$

Tangenten: $\frac{\partial \mathbf{r}}{\partial u}, \frac{\partial \mathbf{r}}{\partial v}$

Normalvektor: $\mathbf{N} = \pm \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}$ ($\pm$, upp/ner)

Einheitsnormal: $\hat{\mathbf{N}} = \mathbf{N}/|\mathbf{N}|$

$$dS = \left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right| du dv$$

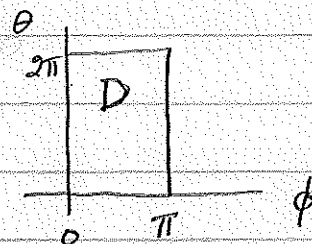
$$d\mathbf{S} = \hat{\mathbf{N}} dS = \pm \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} du dv$$

$$\iint_{\mathcal{S}} f dS; \quad \iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} = \iint_{\mathcal{S}} \mathbf{F} \cdot \hat{\mathbf{N}} dS$$

Exempel Sphär $x^2 + y^2 + z^2 = R^2$

Sfäriska koordinater ($\rho = R$):

$$\begin{cases} x = R \sin \phi \cos \theta, \\ y = R \sin \phi \sin \theta, \\ z = R \cos \phi, \end{cases} \quad \begin{aligned} \phi &\in [0, \pi] \\ \theta &\in [0, 2\pi] \end{aligned}$$



$$\begin{cases} \frac{\partial \mathbf{r}}{\partial \phi} = R \cos \phi \cos \theta \mathbf{i} + R \cos \phi \sin \theta \mathbf{j} - R \sin \phi \mathbf{k} \\ \frac{\partial \mathbf{r}}{\partial \theta} = -R \sin \phi \sin \theta \mathbf{i} + R \sin \phi \cos \theta \mathbf{j} \end{cases}$$

$$\frac{\partial \mathbf{r}}{\partial \phi} \times \frac{\partial \mathbf{r}}{\partial \theta} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ R \cos \phi \cos \theta & R \cos \phi \sin \theta & -R \sin \phi \\ -R \sin \phi \sin \theta & R \sin \phi \cos \theta & 0 \end{vmatrix} =$$

$$= R^2 \sin \phi \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos \phi \cos \theta & \cos \phi \sin \theta & -\sin \phi \\ -\sin \theta & \cos \theta & 0 \end{vmatrix} =$$

$$= R^2 \sin \phi \left(\sin \phi \cos \theta \mathbf{i} + \sin \phi \sin \theta \mathbf{j} + \cos \phi \mathbf{k} \right)$$

$$= \frac{\mathbf{r}}{R} = \frac{\mathbf{r}}{|\mathbf{r}|} = \hat{\mathbf{r}}$$

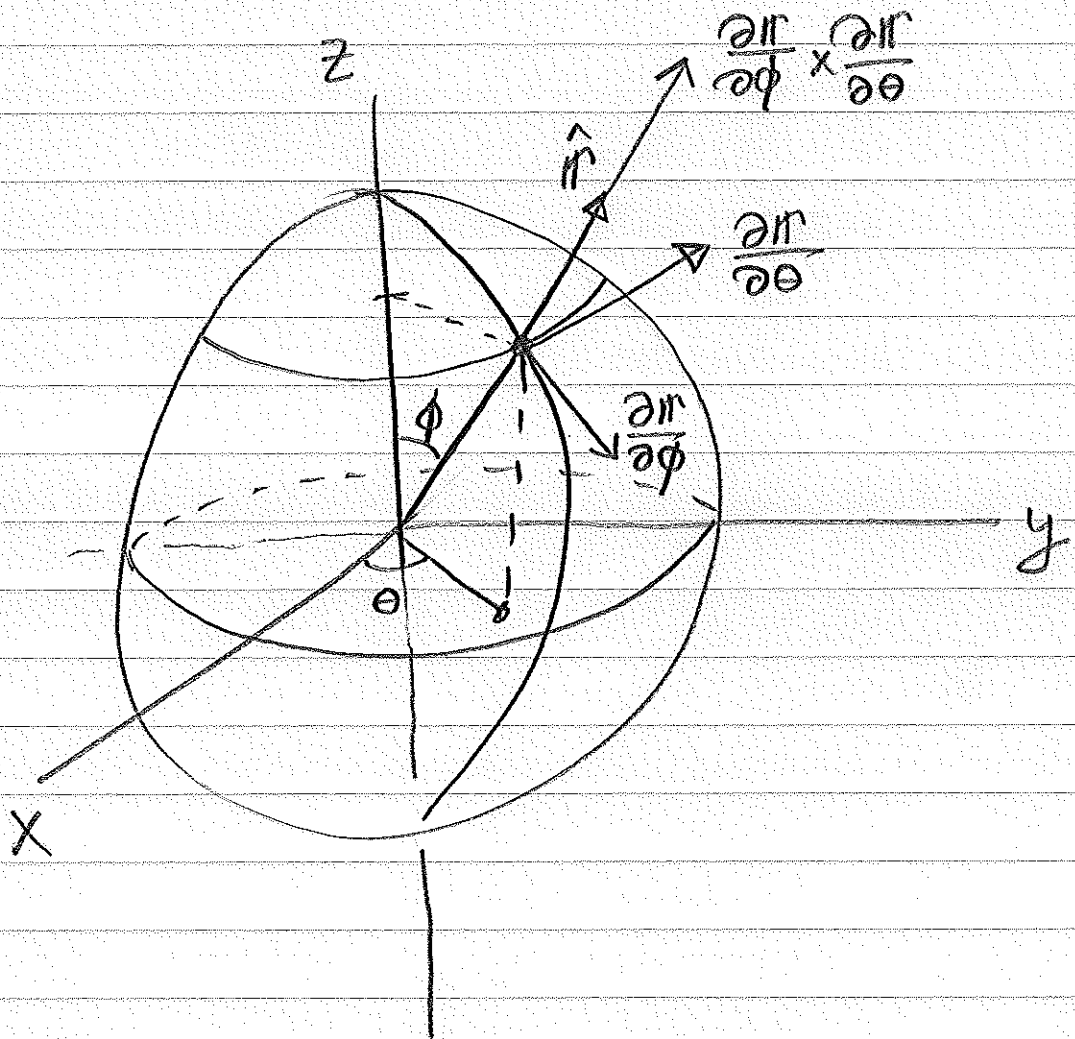
$$= R^2 \sin \phi \hat{\mathbf{r}}$$

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$$\left| \frac{\partial \mathbf{r}}{\partial \phi} \times \frac{\partial \mathbf{r}}{\partial \theta} \right| = R^2 \sin \phi \quad \left(\sin \phi \geq 0 \right) \\ \text{for } \phi \in [0, \pi]$$

$$dS = R^2 \sin \phi \, d\phi \, d\theta \quad \mathbf{N} = \pm R^2 \sin \phi \, \hat{\mathbf{r}}$$

$$d\mathbf{S} = \pm R^2 \sin \phi \, \hat{\mathbf{r}} \, d\phi \, d\theta \quad \hat{\mathbf{N}} = \pm \hat{\mathbf{r}}$$



Surface area: $A = \iint_{\mathcal{S}} dS = \iint_D R^2 \sin \phi \, d\phi \, d\theta$

$$= \{\text{rectangle}\} = R^2 \int_0^{2\pi} \int_0^{\pi} \sin \phi \, d\phi \, d\theta =$$

$$= R^2 \int_0^{2\pi} d\theta \int_0^{\pi} \sin \phi \, d\phi = R^2 \cdot 2\pi \cdot 2 = 4\pi R^2$$

Flödet av $F(r) = \frac{r}{|r|^3}$ ut genom S :

$$\iint_S F \cdot \hat{N} dS = \iint_D \underbrace{\frac{r}{|r|^3} \cdot \begin{pmatrix} + \\ (-) \end{pmatrix} \hat{r}}_{\substack{\text{ut} \\ = \hat{N}}} \underbrace{R^2 \sin \phi}_{dS} d\phi d\theta$$

$$= \iint_D \underbrace{\frac{r \cdot \hat{r}}{|r|^3}}_{= \frac{R}{R^3}} R^2 \sin \phi d\phi d\theta$$

$$= \iint_D \sin \phi d\phi d\theta = \int_0^{2\pi} d\theta \int_0^\pi \sin \phi d\phi = 4\pi$$

Grad, div, rot 16.1Med nabla-operatorn

$$\nabla = \frac{\partial}{\partial x} \mathbf{i} + \frac{\partial}{\partial y} \mathbf{j} + \frac{\partial}{\partial z} \mathbf{k} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$$

kan vi bilda

gradienten av skalärt fält:

$$\nabla \phi = \text{grad } \phi = \frac{\partial \phi}{\partial x} \mathbf{i} + \frac{\partial \phi}{\partial y} \mathbf{j} + \frac{\partial \phi}{\partial z} \mathbf{k}$$

och divergensen och rotationen

av vektorfält:

$$\nabla \cdot \mathbf{F} = \text{div } \mathbf{F} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot (F_1, F_2, F_3)$$

$$= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$\nabla \times \mathbf{F} = \text{rot } \mathbf{F} = \text{curl } \mathbf{F} =$$

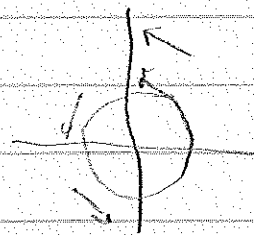
$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)$$

Exempel Helkroppsrotation

$$\mathbf{v} = \Omega (-y \mathbf{i} + x \mathbf{j})$$

$$\nabla \cdot \mathbf{v} = \Omega \left(\frac{\partial}{\partial x}(-y) + \frac{\partial}{\partial y}x \right) = 0$$

$$\nabla \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\Omega y & \Omega x & 0 \end{vmatrix} = 2\Omega \mathbf{k}$$

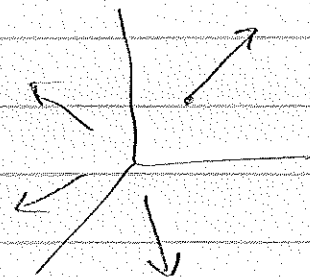


Ingen divergens, bara rotation.

Exempel $\mathbf{F}(\mathbf{r}) = \mathbf{r}$

$$\nabla \cdot \mathbf{F} = 1 + 1 + 1 = 3$$

$$\nabla \times \mathbf{F} = \mathbf{0} = \mathbf{0}$$



Divergerar, men roterar ej.

Exempel

$$\mathbf{F} = \frac{\mathbf{r}}{|\mathbf{r}|^3} = \frac{\mathbf{r}}{r^3} = \frac{(x, y, z)}{r^3}$$

$$\frac{\partial F_1}{\partial x} = \frac{\partial}{\partial x} \left(\frac{x}{r^3} \right) = \frac{\frac{\partial x}{\partial x} r^3 - x \frac{\partial r^3}{\partial x}}{r^6} =$$

$$= \frac{r^3 - x \cdot 3r^2 \frac{\partial r}{\partial x}}{r^6} = \frac{r^3 - 3 \frac{x^2}{r}}{r^6} = \frac{r^2 - 3x^2}{r^5}$$

På samma vis:

$$\frac{\partial F_2}{\partial y} = \frac{r^2 - 3y^2}{r^5}$$

$$\frac{\partial F_3}{\partial z} = \frac{r^2 - 3z^2}{r^5}$$

$$\begin{aligned} r &= \sqrt{x^2 + y^2 + z^2} \\ \frac{\partial r}{\partial x} &= \frac{2x}{2\sqrt{\dots}} = \frac{x}{r} \end{aligned}$$

$$\text{Så att } \nabla \cdot \mathbf{F} = \frac{3r^2 - 3(x^2 + y^2 + z^2)}{r^5} = 0, r \neq 0$$

Divergensfritt utom i origo.

Vi har redan sett att flödet ut genom sfären är

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = 4\pi$$

Vi ska återkomma till detta exempel senare.

Räkneregler 16.2

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Sats 3

a) $\nabla(\phi\psi) = \nabla\phi\psi + \phi\nabla\psi$

b) $\nabla\cdot(\phi\mathbb{F}) = \nabla\phi\cdot\mathbb{F} + \phi\nabla\cdot\mathbb{F}$

c) d) e) f) i) läs boken

g) $\nabla\cdot(\nabla\times\mathbb{F}) = 0$

h) $\nabla\times\nabla\phi = 0$

Gör bevis av a, b, c, d, g, h
Resten utan bevis.

Obs: $\mathbb{F} = \nabla\phi \stackrel{(h)}{\implies} \nabla\times\mathbb{F} = 0$

Alltså: ett konservativt fält
är rotationsfritt. Detta är
det nödvändiga villkoret i 15.2

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Det är också tillräckligt
i enkelt sammanhängande område
enligt följande sats.

Sats 4 Antag: 1) $\nabla \times \mathbb{F} = 0$ i D

2) D är enkelt samman-
hängande.

Då är $\mathbb{F}$ konservativt i D ,
dvs det finns ϕ så att $\mathbb{F} = \nabla \phi$
i D .

Det betyder att ekv. systemet

$$\begin{cases} \frac{\partial \phi}{\partial x} = F_1 \\ \frac{\partial \phi}{\partial y} = F_2 \\ \frac{\partial \phi}{\partial z} = F_3 \end{cases} \quad \text{är lösbart i } D.$$

Exempel

$F = r$. Då är $\nabla \times F = 0 = \mathbf{0}$.

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Vi löser

$$\begin{cases} \frac{\partial \phi}{\partial x} = x \\ \frac{\partial \phi}{\partial y} = y \\ \frac{\partial \phi}{\partial z} = z \end{cases} \Rightarrow \begin{cases} \phi(x, y, z) = \frac{1}{2}x^2 + f(y, z) \\ \phi(x, y, z) = \frac{1}{2}y^2 + g(x, z) \\ \phi(x, y, z) = \frac{1}{2}z^2 + h(x, y) \end{cases}$$

Välj $f(y, z) = \frac{1}{2}(y^2 + z^2) + C$

$$g(x, z) = \frac{1}{2}(x^2 + z^2) + C$$

$$h(x, y) = \frac{1}{2}(x^2 + y^2) + C$$

Då fås

$$\phi(x, y, z) = \frac{1}{2}(x^2 + y^2 + z^2) + C$$