Fö 8.1 Sammanfattning

Alla dessa integraler.

1. Envariabelintegral: $\int_I f\,dx = \int_a^b f(x)\,dx = \lim \sum_{i=1}^{N} f(x_i)\Delta x_i$

Variabelbyte: $x = g(u)$, $dx = g'(u)\,du$

$$\int_a^b f(x)\,dx = \int_{g(a)}^{g(b)} f(g(u))\,g'(u)\,du$$

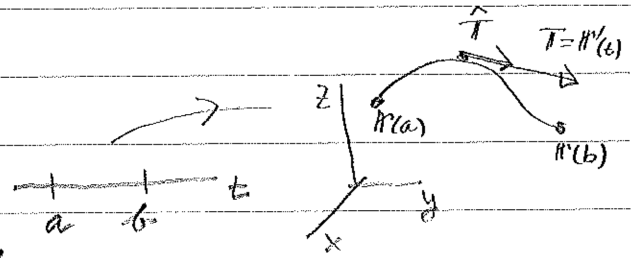
Fundamentalsatsen: $\int_a^b Df(x)\,dx = \left[f(x)\right]_a^b$

Kurvintegral: $\mathbb{r} = \mathbb{r}(t)$, $t \in [a, b]$

$$ds = |\mathbb{r}'(t)|\,dt$$

$$\int_{\mathcal{C}} f\,ds = \int_a^b f(\mathbb{r}(t))\,|\mathbb{r}'(t)|\,dt$$

Längd: $L = \int_{\mathcal{C}} ds$



Tangentkurvintegral:

$$d\mathbb{r} = \mathbb{r}'(t)\,dt = \frac{\mathbb{r}'(t)}{|\mathbb{r}'(t)|}\,|\mathbb{r}'(t)|\,dt = \hat{\mathbb{T}}\,ds$$

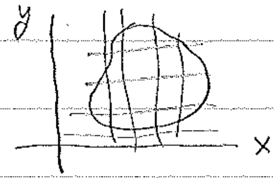
$$\int_{\mathcal{C}} \mathbb{F} \cdot d\mathbb{r} = \int_{\mathcal{C}} \mathbb{F} \cdot \hat{\mathbb{T}}\,ds = \int_a^b \mathbb{F}(\mathbb{r}(t)) \cdot \mathbb{r}'(t)\,dt$$

Tolkning: arbete.

Fundamentalsats: om $\mathbb{F} = \nabla\phi$ så är

$$\int_{\mathcal{C}} \mathbb{F} \cdot d\mathbb{r} = \int_{\mathcal{C}} \nabla\phi \cdot d\mathbb{r} = \phi(\mathbb{r}(b)) - \phi(\mathbb{r}(a))$$

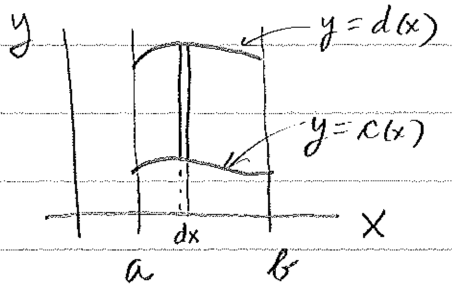
2. Dubbelintegral: $\displaystyle\iint_R f\,dA = \lim \sum_{i=1}^{N}\sum_{j=1}^{M} f(x_i, y_j)\,\Delta x_i \Delta y_j$



$dA = dx\,dy$

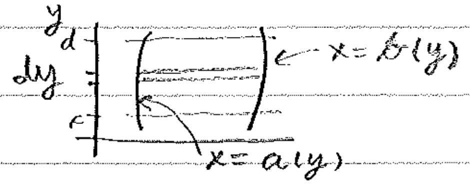
Upprepad integration:

$$\iint_R f\,dA = \int_a^b \int_{c(x)}^{d(x)} f(x,y)\,dy\,dx$$



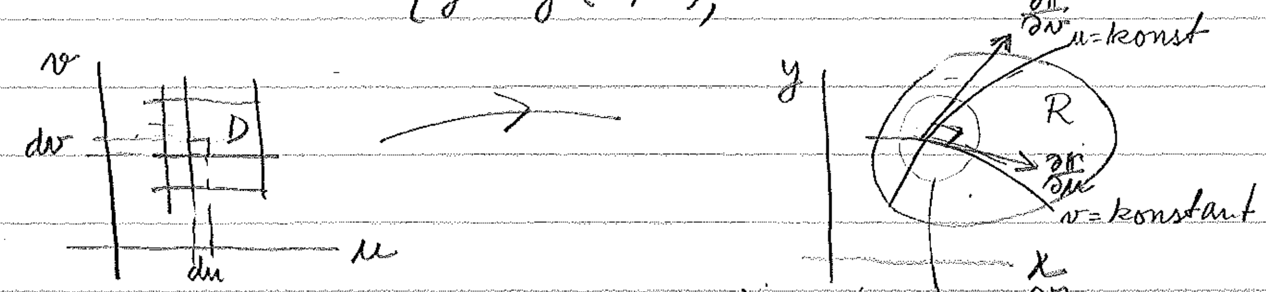
eller

$$\iint_R f\,dA = \int_c^d \int_{a(y)}^{b(y)} f(x,y)\,dx\,dy$$

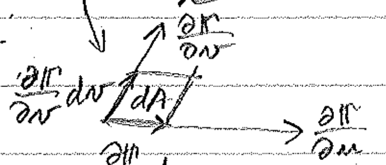


Area: $A = \iint_R dA$

Variabelbyte: $\begin{cases} x = x(u,v), \\ y = y(u,v), \end{cases}$ $(u,v) \in D$, $\mathbf{r} = \mathbf{r}(u,v)$



$$dA = \left|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right| du\,dv = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$



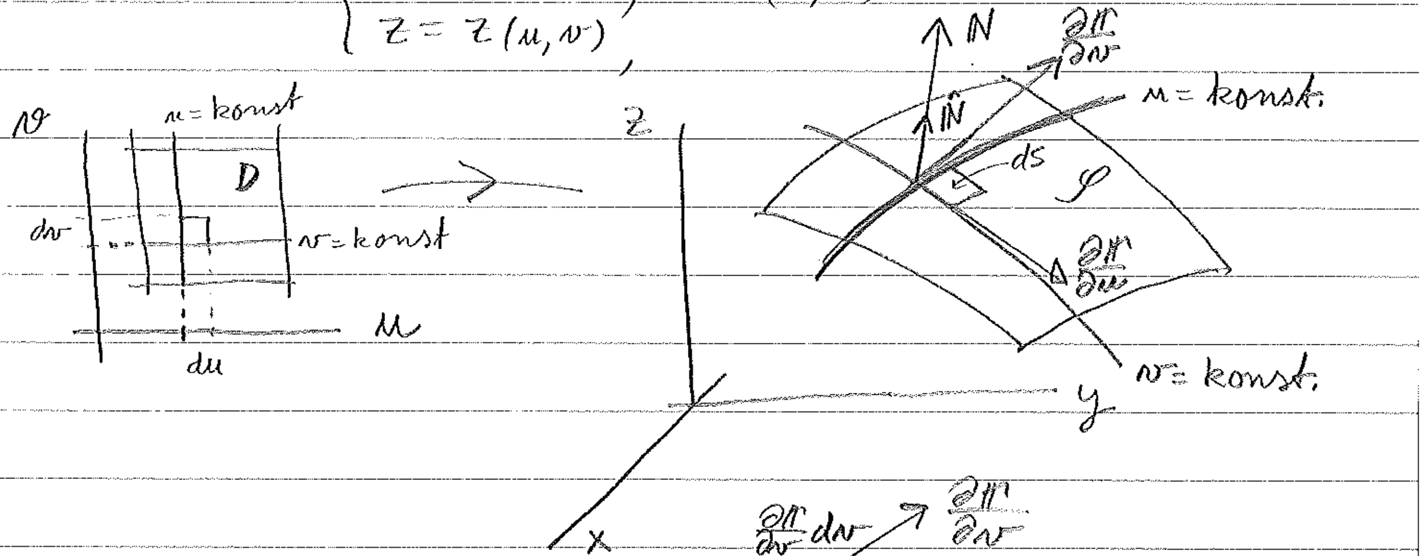
Koordinatkurvor, tangenter, Jacobideterminant.

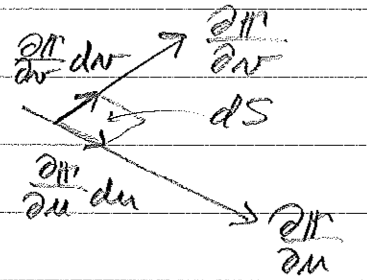
$$\iint_R f\,dA = \iint_D f(\mathbf{r}(u,v)) \left|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right| du\,dv$$

$$= \iint_D f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

Yta: $\mathcal{S}: \mathbb{r} = \mathbb{r}(u,v)$, $(u,v) \in D$

$$\mathcal{S}: \begin{cases} x = x(u,v), \\ y = y(u,v), \\ z = z(u,v), \end{cases} \quad (u,v) \in D$$





Koordinatkurvor.

Tangenter: $\frac{\partial \mathbb{r}}{\partial u}, \frac{\partial \mathbb{r}}{\partial v}$

Normalvektor: $\mathbb{N} = \pm \frac{\partial \mathbb{r}}{\partial u} \times \frac{\partial \mathbb{r}}{\partial v}$

Enhetsnormal: $\hat{\mathbb{N}} = \frac{\mathbb{N}}{|\mathbb{N}|}$

$$dS = |\mathbb{N}|\, du\, dv = \left| \frac{\partial \mathbb{r}}{\partial u} \times \frac{\partial \mathbb{r}}{\partial v} \right| du\, dv$$

Ytintegral: $\iint_{\mathcal{S}} f\, dS = \iint_D f(\mathbb{r}(u,v)) \left| \frac{\partial \mathbb{r}}{\partial u} \times \frac{\partial \mathbb{r}}{\partial v} \right| du\, dv$

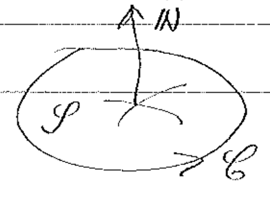
Area: $A = \iint_{\mathcal{S}} dS$

Flödesintegral (normalytintegral):

$$\iint_{\mathcal{S}} \mathbb{F} \cdot d\mathbb{S} = \iint_{\mathcal{S}} \mathbb{F} \cdot \hat{\mathbb{N}}\, dS = \pm \iint_D \mathbb{F}(\mathbb{r}(u,v)) \cdot \left(\frac{\partial \mathbb{r}}{\partial u} \times \frac{\partial \mathbb{r}}{\partial v} \right) du\, dv$$

Fundamentalsats: Stokes

$$\iint_{\mathcal{S}} (\nabla \times \mathbb{F}) \cdot \hat{\mathbb{N}}\, dS = \int_{\mathcal{C}} \mathbb{F} \cdot d\mathbb{r}$$



3. Trippelintegral

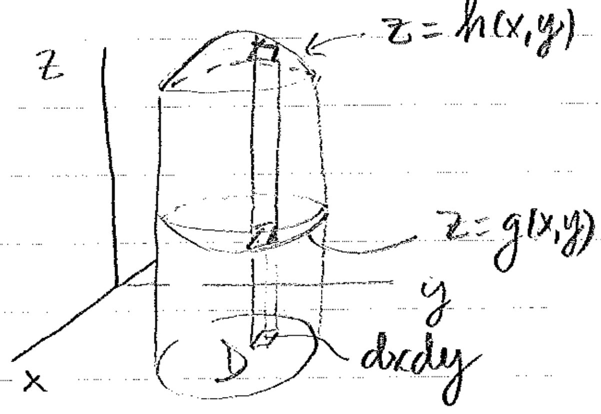
$$\iiint_R f\,dV = \lim \sum_i \sum_j \sum_k f(x_i, y_j, z_k)\,\Delta x_i\,\Delta y_j\,\Delta z_k$$

$dV = dx\,dy\,dz$ volymselement

Upprepad integration:

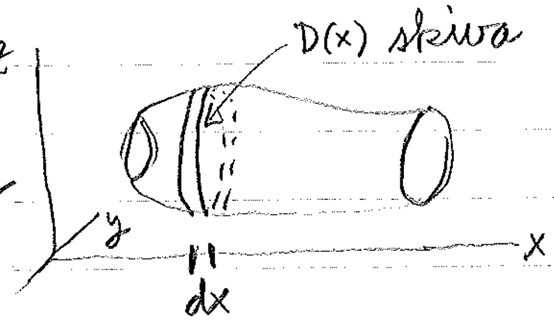
1) Renkelt i z, dvs mellan två grafer $g(x,y) \leq z \leq h(x,y)$, $(x,y) \in D$

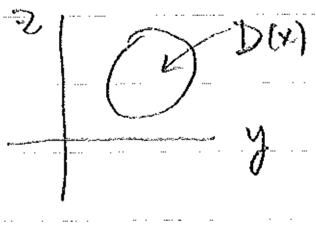
$$\iiint_R f\,dV = \iint_D \left(\int_{g(x,y)}^{h(x,y)} f(x,y,z)\,dz \right) dx\,dy$$



2) skivmetoden

$$\iiint_R f\,dV = \int_a^b \left(\iint_{D(x)} f(x,y,z)\,dy\,dz \right) dx$$





Volym : $V = \iiint_R dV$

Tyngdpunkt: $\bar{x} = \frac{1}{V} \iiint_R x \, dV$

Variabelbyte: $\mathbf{r} = \mathbf{r}(u, v, w)$, $(u, v, w) \in D$

Koordinatkurvor,
Tangenter $\frac{\partial \mathbf{r}}{\partial u}, \frac{\partial \mathbf{r}}{\partial v}, \frac{\partial \mathbf{r}}{\partial w}$.
Jacobideterminant.

$$dV = \left| \left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right) \cdot \frac{\partial \mathbf{r}}{\partial w} \right| du\, dv\, dw$$

$$= \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| du\, dv\, dw$$

$$\iiint_R f \, dV = \iiint_D f(\mathbf{r}(u, v, w)) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| du\, dv\, dw$$

Fundamentalsats: Gauss

$$\iiint_R \nabla \cdot \mathbf{F} \, dV = \iint_S \mathbf{F} \cdot \hat{\mathbf{N}} \, dS$$

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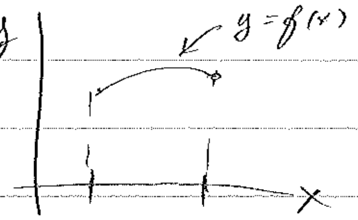
Exempel Graf. $y = f(x)$, $x \in [a, b]$

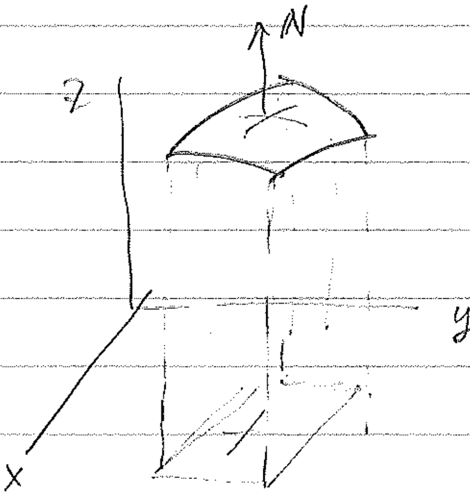


$$\begin{cases} x = t \\ y = f(t), \quad t \in [a, b] \end{cases}$$

$$\mathbf{r}'(t) = \mathbf{i} + f'(t)\,\mathbf{j}$$

$$ds = \sqrt{1 + f'(t)^2}\, dt$$

Exempel Graf. $z = f(x, y)$, $x, y \in D$



$$\begin{cases} x = u \\ y = v \\ z = f(u, v) \end{cases}$$

$$\frac{\partial \mathbf{r}}{\partial u} = \mathbf{i} + f'_u \mathbf{k}$$

$$\frac{\partial \mathbf{r}}{\partial v} = \mathbf{j} + f'_v \mathbf{k}$$

$$\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & f'_u \\ 0 & 1 & f'_v \end{vmatrix} = -f'_u \mathbf{i} - f'_v \mathbf{j} + \mathbf{k}$$

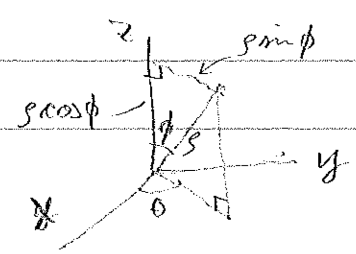
$$dS = \sqrt{1 + {f'_u}^2 + {f'_v}^2}\, du\, dv$$

$$d\mathbf{S} = \hat{\mathbf{N}}\, dS = \pm \left(-f'_u \mathbf{i} - f'_v \mathbf{j} + \mathbf{k} \right) du\, dv$$

Viktiga koordinater

Polära (r, θ) $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

Cylindriska (r, θ, z) $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$



Sfäriska. (ρ, ϕ, θ) $\begin{cases} x = \rho\sin\phi\cos\theta \\ y = \rho\sin\phi\sin\theta \\ z = \rho\cos\phi \end{cases}$