FÖ 8.2 Sammanfattning.

+ tenta 08-05-26 nr 6,7

Alla dessa derivator

Partiell derivata

$$z = f(x,y)$$

$$\frac{\partial f}{\partial x} = f'_x = f'_1 = \frac{\partial z}{\partial x} \quad \text{osv.}$$

(skriv alltid "prim": $f'_x = f'_1$)

Kedjeregeln

t. ex. $g(x,y,t) = f(u(x,y), v(x,t))$

$$\frac{\partial g}{\partial y} = f'_1(u(x,y), v(x,t))\, u'_1(x,y) +$$

$$+ f'_2(u(x,y), v(x,t))\, v'_1(x,t)$$

$$\frac{\partial g}{\partial t} = f'_2(u(x,y), v(x,t))\, v'_2(x,t)$$

Riktig derivata ges av linjäriseringen

$$L(x,y) = f(a,b) + f'_x(a,b)(x-a) + f'_y(a,b)(y-b)$$

$$= f(a,b) + f'_x(a,b)\, h + f'_y(a,b)\, k$$

f är deriverbar i (a,b) om

$$\lim \frac{f(x,y) - L(x,y)}{\sqrt{h^2+k^2}} \to 0$$

Då gäller:

$$f(x,y) \approx L(x,y) \quad \text{nära } (a,b)$$

Högre ordning: $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$

På matrisform

$y = f(x)$

$f : \mathbb{R}^m \to \mathbb{R}$, $\underset{h = x-a}{x \in \mathbb{R}^m}$, $y \in \mathbb{R}$

$$L(x) = f(a) + f'(a)(x-a)$$

$$= f(a) + f'(a)h$$

$$= [\cdot] + [\cdot \; \cdot \; \cdot] \begin{bmatrix} \cdot \\ \cdot \\ \cdot \end{bmatrix}$$

Jacobi-matris:

$$f'(a) = Df(a) = [f_1'(a), \ldots, f_2'(a)]$$

$f : \mathbb{R}^m \to \mathbb{R}^n$ $\quad y = f(x), x \in \mathbb{R}^m, y \in \mathbb{R}^n$

$$L(x) = f(a) + f'(a)h$$

$$= \underset{n \times 1}{\begin{bmatrix} \cdot \\ \cdot \\ \cdot \end{bmatrix}} + \underset{n \times m}{\begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}} \underset{m \times 1}{\begin{bmatrix} \cdot \\ \cdot \\ \cdot \end{bmatrix}}$$

Kedjeregeln

$$Df(g(x)) = Df(g(x))\, Dg(x)$$

Taylors formel (grad 2)

$$f: \mathbb{R}^m \to \mathbb{R}$$

$$f(x) = f(a) + f'(a)h + \frac{1}{2} h^T f''(a) h + R$$

Hesse-matrisen: $D^2 f(a) = f''(a) = \begin{bmatrix} f''_{11} & \cdots & f''_{1m} \\ \vdots & & \\ f''_{m1} & \cdots & f''_{mm} \end{bmatrix}$ symmetrisk

Newtons metod

Lös ekv. systemet

$$f(x) = 0$$

där $f: \mathbb{R}^n \to \mathbb{R}^m$.

Antag att x_0 är en approx lösn. Vi vill hitta en bättre approx $x = x_0 + h$.

Lös den linjäriserade ekv

$$L(x) = 0$$

dvs $f(x_0) + f'(x_0)h = 0$

$$f'(x_0)h = -f(x_0)$$

Detta leder till algoritmen

Välj x och $h = tol + 1$

while $|h| > tol$

    beräkna residualen $b = -f(x)$
    beräkna J-matrisen $A = f'(x)$
    lös linj. ekv. $Ah = b$
    uppdatera $x = x + h$

end

Lokala max-, min-, sadelpunkter

1. kritisk punkt ges av

$$f'(x)^T = 0$$

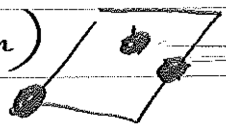
dvs
$$\begin{cases} f'_1(x) = 0 \\ f'_2(x) = 0 \\ \vdots \\ f'_n(x) = 0 \end{cases}$$

2. ~~~~ undersök Hesse-matrisen

pos. def.	lok min
neg. def.	lok max
indefinit	sadel
annars	ingen info

Max/min problem

1. Inre singulära punkter $f'(x)$ existerar ej
   Inre kritiska punkter $f'(x) = 0$

2. Randpunkter (extrempunkter på randen) 
3. Jämför alla dessa punkter.

Lagranges multiplikatormetod: Om x_0 är extrempunkt till $f(x)$ med bivillkor $g(x)=0$, så är (x_0, λ_0) kritisk punkt till $L(x,\lambda) = f(x) + \lambda g(x)$. Dvs lösning till $DL(x,\lambda) = 0$,

$$\begin{cases} \frac{\partial L}{\partial x}(x,\lambda) = f'_x(x) + \lambda g'(x) = 0 \\ \frac{\partial L}{\partial \lambda}(x,\lambda) = g(x) = 0 \end{cases}$$

Matlab!!

Geometriska derivator: grad, div, rot, $\nabla\phi$, $\nabla \cdot \mathbb{F}$, $\nabla \times \mathbb{F}$

Riktnings derivata: $u = u_1 i + u_2 j$
$|u| = 1$

$$D_u f(a,b) = \frac{d}{dt} f(a + t u_1, b + t u_2)\Big|_{t=0}$$

$$D_u f(a,b) = u \cdot \nabla f(a,b)$$

obs: riktningsvektorn u ska normeras!

∇f är normalvektor till nivåkurva och nivåyta.

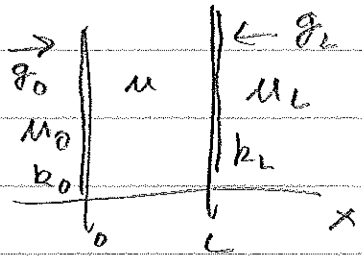
Produktderivator och deriveringsregler

t. ex. $\nabla\cdot(\phi\mathbb{F}) = \nabla\phi\cdot\mathbb{F} + \phi\,\nabla\cdot\mathbb{F}$

och $\nabla\times\nabla\phi = 0$

$\nabla\cdot(\nabla\times\mathbb{F}) = 0$

Randvärdesproblem



1-D
$$\begin{cases} -D(a\,Du) = f & \text{in } (0,L) \\ a\,D_N u + k(u-u_A) = g & x = 0, L \end{cases}$$

$$D_N u = \begin{cases} Du & x = L \\ -Du & x = 0 \end{cases} \qquad k = \begin{cases} k_L \\ k_0 \end{cases} \text{ osv.}$$

$j = -aDu$

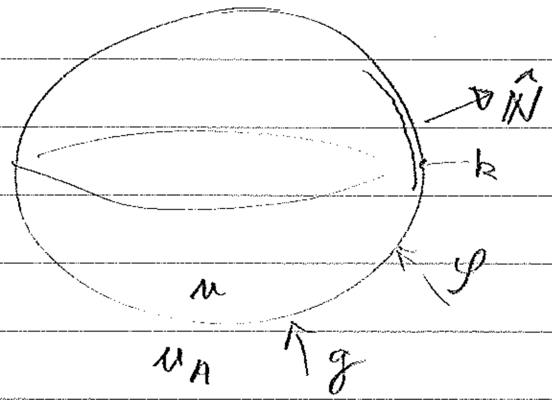
Veta betydelsen

Lösa med två integrationer

Svag form, FEM

3-D

$$\begin{cases} -\nabla\cdot(a\nabla u) = f & \text{i } D \\ u = u_A & \text{på } S_1 \\ a D_{\hat{N}} u + k(u-u_A) = g & \text{på } S_2 \end{cases}$$



$$\mathbb{F} = -a\nabla u$$

Härleda detta.

Randvillkoret

utflöde genom S:

$$\begin{cases} \hat{N}\cdot\mathbb{F} = k(u-u_A) - g \\ \hat{N}\cdot\mathbb{F} = \hat{N}\cdot(-a\nabla u) = -a\,\hat{N}\cdot\nabla u = \end{cases}$$

$$= -a\,D_{\hat{N}} u.$$

$$\Rightarrow \quad -a\,D_{\hat{N}} u = k(u-u_A) - g$$

$k=\infty$: $\quad \underbrace{\frac{1}{k}}_{\to 0} a\,D_{\hat{N}} + u - u_A = \underbrace{\frac{1}{k}}_{\to 0} g$ på S_1

$$u - u_A = 0 \quad \text{på } S_1$$

Tenta-problem.

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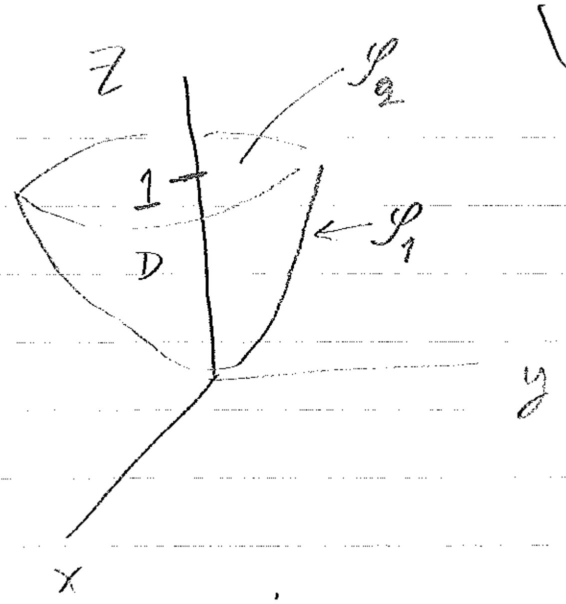


$\mathbb{F} = (x, y, z^2)$

$D = \{(x,y,z) : x^2+y^2 \le z,\ 0 \le z \le 1\}$

$S = S_1 \cup S_2$.

Beräkna flödet ut ur D med hjälp av Gauss sats.

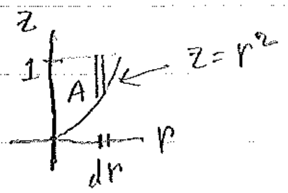
$\nabla \cdot \mathbb{F} = 1+1+2z = 2(1+z)$.

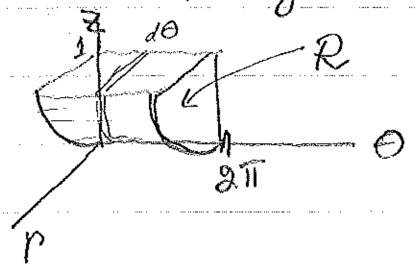
Gauss: $\iint_S \mathbb{F} \cdot \hat{N} dS = \iiint_D \nabla \cdot \mathbb{F}\, dV = \iiint_D 2(1+z)\,dx\,dy\,dz$

Trippelintegral ska beräknas.

1) Cylinderkoordinater (r, θ, z).

Området D: $r^2 = x^2+y^2 \le z$, $0 \le z \le 1$, $0 \le \theta \le 2\pi$





$$\iiint_D 2(1+z)\,dx\,dy\,dz = \iiint_R 2(1+z)\, r\,dr\,d\theta\,dz = \left\{\begin{matrix}\text{skiv-} \\ \text{metod}\end{matrix}\right\}$$

trippelintegral

$$= \int_0^{2\pi} \Big(\underbrace{\iint 2(1+z) r\, dr\,dz}_{A(\theta) = A \text{ beror ej på } \theta,} \Big) d\theta = 2\pi \iint_A 2(1+z)\, r\,dr\,dz =$$

dubbelint.

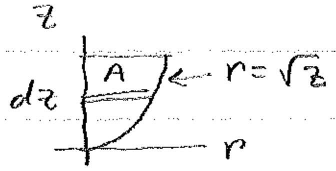
$$\stackrel{\downarrow}{=} 4\pi \int_0^1 \Big(\int_{r^2}^1 (1+z) r\,dz \Big) dr = 4\pi \int_0^1 r \left[z + \tfrac{1}{2} z^2 \right]_{r^2}^1 dr =$$

$$= 4\pi \int_0^1 r\left(1+\frac{1}{2} - r^2 - \frac{1}{2} r^4\right) dr$$

$$= 4\pi \int_0^1 \left(\frac{3}{2} r - r^3 - \frac{1}{2} r^5\right) dr$$

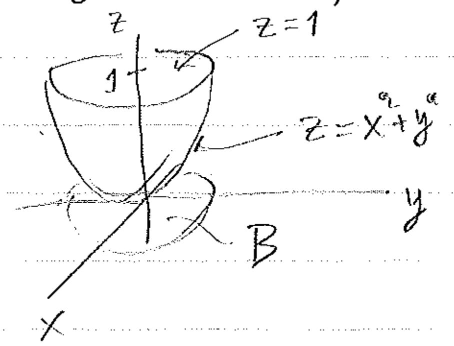
$$= 4\pi \left(\frac{3}{4} - \frac{1}{4} - \frac{1}{12}\right) = 4\pi \, \frac{5}{12} = \frac{5}{3}\pi$$

eller



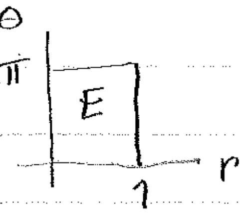
$$4\pi \int_0^1 \left(\int_0^{\sqrt{z}} (1+z)\, r \, dr \right) dz = 4\pi \int_0^1 (1+z) \left[\frac{r^2}{2} \right]_0^{\sqrt{z}} dz$$

2) D är enkelt i z: $x^2+y^2 \le z \le 1$, $(x,y) \in B$



$$\iiint_D 2(1+z)\, dx\, dy\, dz =$$

$$= \iint_B \left(\int_{x^2+y^2}^{1} 2(1+z)\, dz \right) dx\, dy =$$



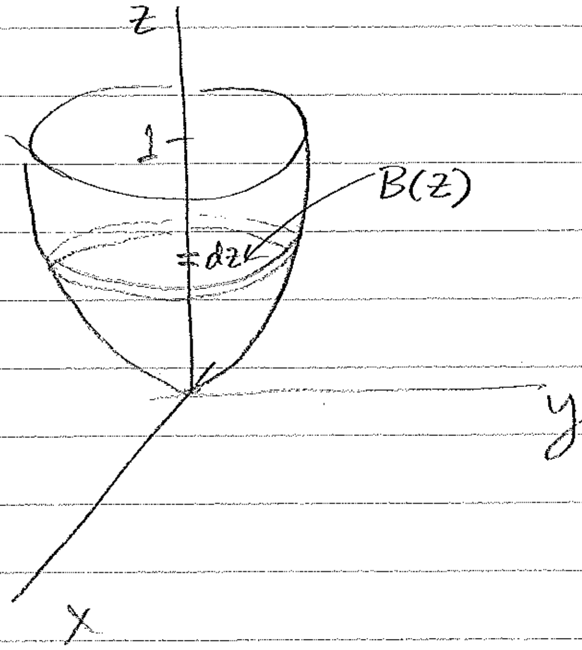
$$= \iint_B \left[2\left(z + \frac{1}{2} z^2\right) \right]_{x^2+y^2}^{1} dx\, dy$$

$$= 2 \iint_B \left(\frac{3}{2} - (x^2+y^2) - \frac{1}{2}(x^2+y^2)^2 \right) dx\, dy = \{\text{poläre}\} =$$

$$= 2 \iint_E \left(\frac{3}{2} - r^2 - \frac{1}{2} r^4 \right) r\, dr\, d\theta = \left\{ \begin{array}{l} E \text{ är} \\ \text{rektangel} \end{array} \right\}$$

$$= 2 \int_0^{2\pi} \left(\int_0^1 \left(\frac{3}{2} r - r^3 - \frac{1}{2} r^5 \right) dr \right) d\theta = 4\pi \cdot \frac{5}{12}.$$

3) Skivmetoden.



$$\iiint_D 2(1+z)\,dx\,dy\,dz =$$

$$= \int_0^1 \Big(\iint_{B(z)} 2(1+z)\,dx\,dy \Big)\,dz =$$

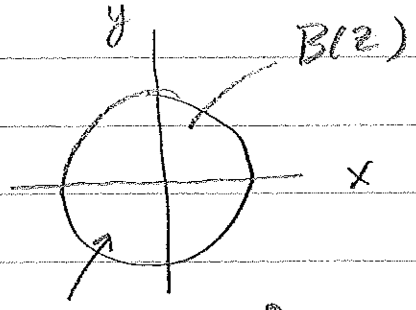
$$= \{ \text{ area}(B(z)) = \pi r^2 = \pi z \}$$

$$= 2\int_0^1 \Big((1+z) \iint_{B(z)} dx\,dy \Big)\,dz$$

$$= 2\int_0^1 (1+z)\,\pi z\,dz$$

$$= 2\pi \int_0^1 (z+z^2)\,dz$$

$$= 2\pi\left(\frac{1}{2}+\frac{1}{3}\right) = 2\pi\,\frac{5}{6} = \frac{5}{3}\pi.$$



$B(z): \; x^2+y^2 \le r^2 = z$

$\frac{3+2}{6}$

Obs: om man bara vill beräkna flödet genom $\mathcal{S}_1$ med Gauss så får man först sluta ytan genom att lägga till $\mathcal{S}_2$.

$$\iint_{\mathcal{S}_1} \mathbb{F} \cdot \hat{\mathbb{N}}\, dS = \iint_{\mathcal{S}} \mathbb{F} \cdot \hat{\mathbb{N}}\, dS - \iint_{\mathcal{S}_2} \mathbb{F} \cdot \hat{\mathbb{N}}\, dS =$$

$$= \iiint_D \nabla \cdot \mathbb{F}\, dV - \iint_{\mathcal{S}_2} \mathbb{F} \cdot \underbrace{\hat{\mathbb{N}}}_{=\mathbb{k}}\, dS$$

$$= \frac{5}{3}\pi - \iint_{\mathcal{S}_2} \underbrace{F_3}_{=1 \text{ på } \mathcal{S}_2}\, dS = \frac{5}{3}\pi - \operatorname{area}(\mathcal{S}_2)$$

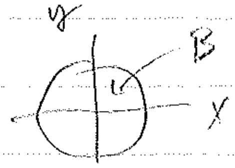
$$= \frac{5}{3}\pi - \pi = \frac{2}{3}\pi.$$

(5

Parametrisera $\mathcal{S}_1$. (för uppg. 6)

Flera sätt.

1) Graf. $z = x^2+y^2$, $(x,y) \in B = \{x^2+y^2 \le 1\}$



$$\begin{cases} x = u \\ y = v \\ z = u^2+v^2 \end{cases}$$

$$\frac{\partial \mathbf{r}}{\partial u} = (1, 0, 2u), \quad \frac{\partial \mathbf{r}}{\partial v} = (0, 1, 2v)$$

$$\mathbb{N} = \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 2u \\ 0 & 1 & 2v \end{vmatrix} = (-2u, -2v, 1)$$

$N_3 = 1 > 0$ uppåt

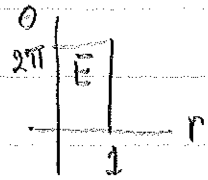
Välj istället $\mathbb{N} = (2u, 2v, -1)$ nedåt
utåt

$$\iint_{\mathcal{S}_1} \mathbb{F} \cdot \hat{\mathbb{N}}\, dS = \iint_B \left(u, v, (u^2+v^2)^2\right) \cdot (2u, 2v, -1)\, du\, dv$$

$$= \iint_B \left(2u^2 + 2v^2 - (u^2+v^2)^2\right) du\, dv \quad \{\text{polära}\}$$

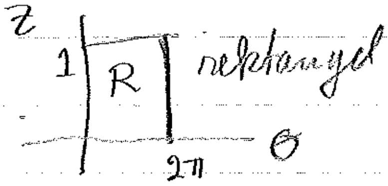
$$= \iint_E (2r^2 - r^4)\, r\, dr\, d\theta = \int_0^{2\pi} d\theta \int_0^1 (2r^3 - r^5)\, dr$$

$$= 2\pi\left(\frac{2}{4} - \frac{1}{6}\right) = \frac{2}{3}\pi$$



2) Cylinderkoord. (θ, z) $\quad r = \sqrt{z}$

$$\begin{cases} x = \sqrt{z}\cos\theta \\ y = \sqrt{z}\sin\theta \\ z = z \end{cases} \qquad (\theta, z) \in [0, 2\pi] \times [0, 1]$$



$$\frac{\partial \mathbf{r}}{\partial \theta} = \ldots, \quad \frac{\partial \mathbf{r}}{\partial z} = \ldots$$

$$\mathbb{N} = \pm \frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial z} = \ldots$$

3) Cylinderkoord (r, θ) $\quad z = r^2$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = r^2 \end{cases} \qquad (r, \theta) \in [0, 1] \times [0, 2\pi]$$

rektangel

$$\frac{\partial \mathbf{r}}{\partial r} = \ldots, \quad \frac{\partial \mathbf{r}}{\partial \theta} = \ldots$$

$$\mathbb{N} = \pm \frac{\partial \mathbf{r}}{\partial r} \times \frac{\partial \mathbf{r}}{\partial \theta} = \ldots$$