

Tidag:

Variabelbyte 14.4

Trippelintegraler 14.5

VariabelbyteNya koordinater (u, v) .

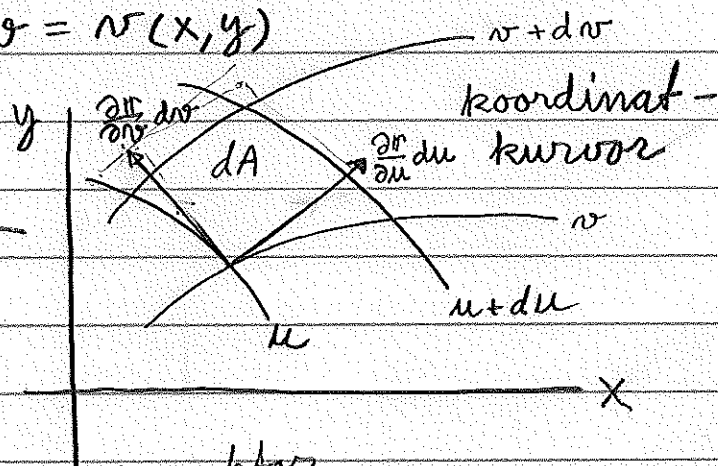
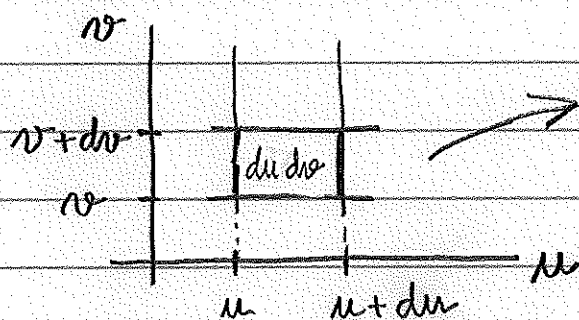
$$\mathbf{r} = \mathbf{r}(u, v)$$

Transformation:

$$\begin{cases} x = x(u, v) \\ y = y(u, v) \end{cases}$$

Invers transformation:

$$\begin{cases} u = u(x, y) \\ v = v(x, y) \end{cases}$$



$$dA = \left| \frac{\partial \mathbf{r}}{\partial u} du \times \frac{\partial \mathbf{r}}{\partial v} dv \right|$$

areaelementet

spänns upp av tangenterna $\frac{\partial \mathbf{r}}{\partial u} du$ och $\frac{\partial \mathbf{r}}{\partial v} dv$.

$$\frac{\partial \mathbf{r}}{\partial u} du \times \frac{\partial \mathbf{r}}{\partial v} dv = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial x}{\partial u} du & \frac{\partial y}{\partial u} du & 0 \\ \frac{\partial x}{\partial v} dv & \frac{\partial y}{\partial v} dv & 0 \end{vmatrix} = \left\{ \begin{array}{l} \text{bryt ut} \\ du \text{ och } dv \end{array} \right\}$$

$$= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} du dv \hat{k} = \frac{\partial(x, y)}{\partial(u, v)} du dv \hat{k}$$

Jacobideterminanten för transformationen:

$$\frac{\partial(x, y)}{\partial(u, v)} = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{pmatrix} = \underbrace{\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}}_{\substack{\text{transponerat } J'(u, v)^T \\ \text{av Jacobimatrisen}}} \quad \left(\det(A^T) = \det(A) \right)$$

$$= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \det(J'(u, v))$$

determinant- = Jacobimatrisen $J'(u, v)$
stretch

Vi får nu

$$dA = \left| \frac{\partial r}{\partial u} du \times \frac{\partial r}{\partial v} dv \right| = \left| \frac{\partial(x, y)}{\partial(u, v)} du dv k \right| \overset{\text{längd}}{=} =$$

$$= \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \overset{\text{abs. belopp}}{=} du dv$$

dvs

$$dx dy = \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv$$

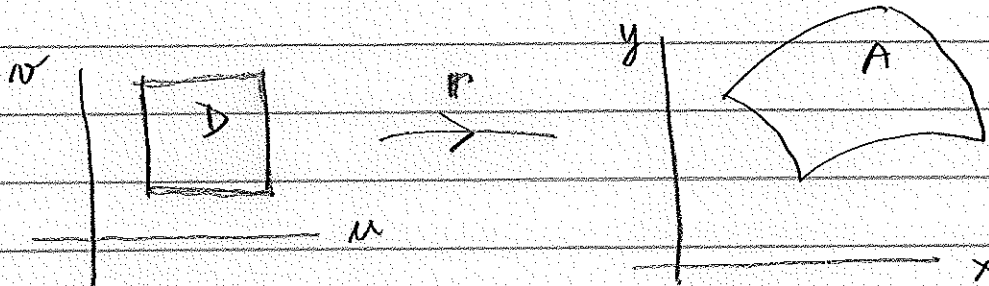
$$= | \det(J'(u, v)) | du dv$$

Obs: $\left| \frac{\partial(x, y)}{\partial(u, v)} \right|$ är yt-skalan vid transformationen.

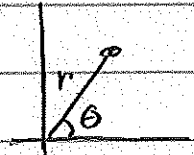
Sats

(Variabelbyte i dubbelintegral) ⁽³⁾

$$\iint_A f(x,y) dx dy = \iint_D f(\pi(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du dv$$



Exempel Polära koord (r, θ) .



$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \end{cases} \quad \begin{cases} r = \sqrt{x^2 + y^2} \\ \theta = \arctan\left(\frac{y}{x}\right) \end{cases}$$

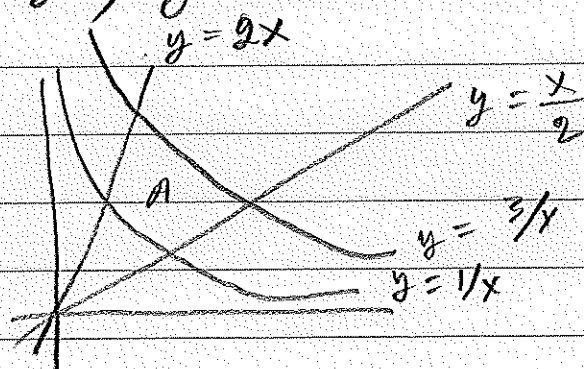
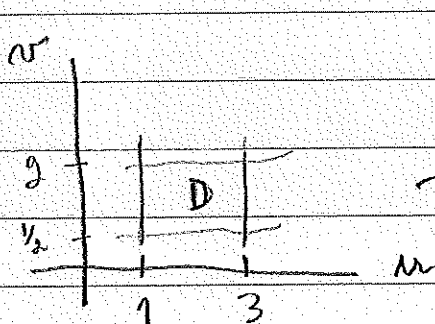
$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos \theta & \sin \theta \\ -r \sin \theta & r \cos \theta \end{vmatrix} = r(\cos^2 \theta + \sin^2 \theta) = r$$

$$dA = \left| \frac{\partial(x,y)}{\partial(r,\theta)} \right| dr d\theta = |r| dr d\theta = r dr d\theta.$$

Exempel

Beräkna $\iint_A \frac{x}{y} dx dy$ där A är området mellan kurvorna

$$y = \frac{1}{x}, \quad y = \frac{3}{x}, \quad y = \frac{x}{2}, \quad y = 2x.$$



Vi har $1 \leq xy \leq 3$, $\frac{1}{2} \leq \frac{y}{x} \leq 2$. Detta bestämmer en rektangel D i uv -planet.

Välj därför $u = xy$, $v = y/x$. Detta är inversa transformationen. Lös ut x, y .

Ekr (2): $y = vx$, i ekr (1): $u = vx^2$, $x = \left(\frac{u}{v}\right)^{1/2}$

Sedan $y = v \sqrt{\frac{u}{v}} = \sqrt{uv}$.

Transformationen blir

$$\begin{cases} x = \sqrt{\frac{u}{v}} \\ y = \sqrt{uv} \end{cases}$$

Jacobideterminanten: $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} =$

$$= \begin{vmatrix} \frac{1}{2\sqrt{uv}} & -\frac{\sqrt{u}}{2\sqrt{v}^3} \\ \frac{\sqrt{v}}{2\sqrt{u}} & \frac{\sqrt{u}}{2\sqrt{v}} \end{vmatrix} = \frac{1}{2v}$$

Integralen bis

$$\iint_A \frac{x}{y} dx dy = \iint_D \frac{\frac{\sqrt{u}}{v}}{\sqrt{uv}} \frac{1}{2v} du dv =$$

$$= \iint_D \frac{1}{2v^2} du dv = \int_1^3 \left(\int_{1/2}^2 \frac{1}{2v^2} dv \right) du$$

$$= \int_1^3 du \int_{1/2}^2 \frac{1}{2v^2} dv = 2 \cdot \left[\frac{1}{-2v} \right]_{1/2}^2$$

$$= 2 \left(\frac{1}{-4} - \frac{1}{-1} \right) = 2 \cdot \frac{3}{4} = \frac{3}{2}$$

Exempel $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$

Bewis Låt $I = \int_{-\infty}^{\infty} e^{-x^2} dx$. Då blir

$$I^2 = \int_{-\infty}^{\infty} e^{-x^2} dx \int_{-\infty}^{\infty} e^{-y^2} dy =$$

$$= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} e^{-x^2} e^{-y^2} dx \right) dy$$

$$= \iint_{\mathbb{R}^2} e^{-x^2-y^2} dx dy$$

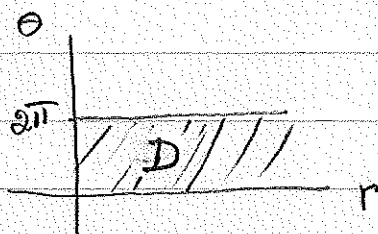
polära
↓

$$= \iint_D e^{-r^2} r dr d\theta = \int_0^{2\pi} \left(\int_0^{\infty} e^{-r^2} r dr \right) d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{\infty} e^{-r^2} r dr$$

(generaliserad integral
positiv integrand
upprepad integration)

$$= 2\pi \left[-\frac{1}{2} e^{-r^2} \right]_0^{\infty} = 2\pi \cdot \frac{1}{2} = \pi$$



Trippelintegralen (14.5)

(7)

$$\iiint_D f \, dV = \iiint_D f(x,y,z) \, dx \, dy \, dz$$

definieras med partition av

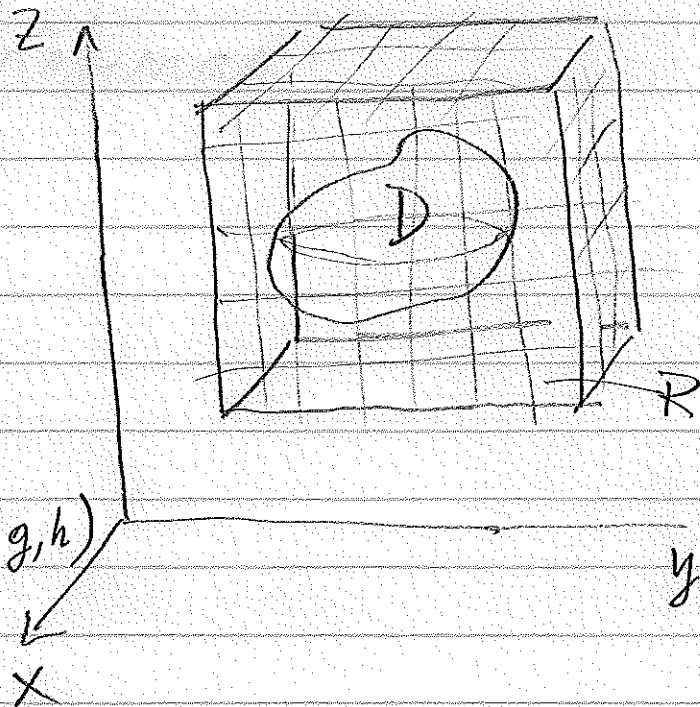
området D och Riemann-summa.

Först över rät-
block R , sedan
allmänt område D

Matlab:

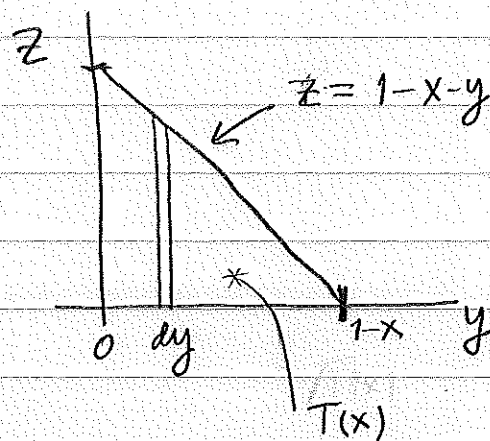
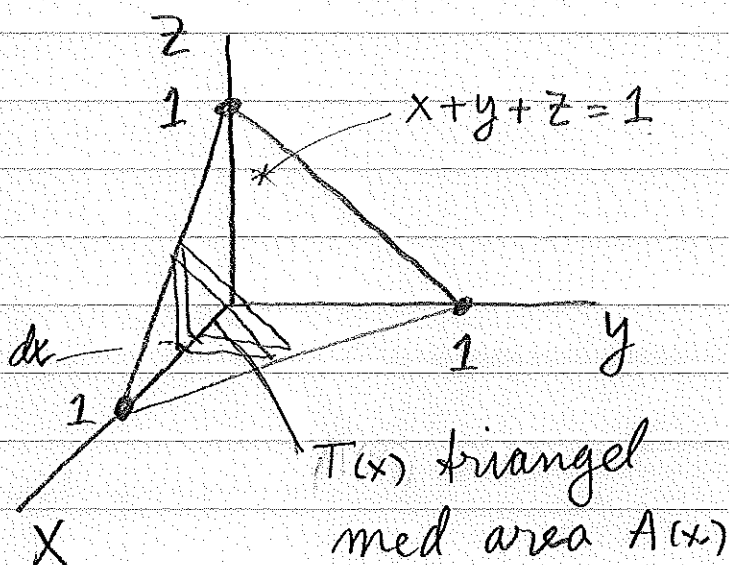
`integral3(f, a, b, c, d, g, h)`

integrerar över
rätblock.



Kan ibland beräknas med
upprepad integration.

Exempel Volymen av tetraeder.



Tetraedern D är enkel i alla riktningar. 1) Skivmetoden:

$$V = \iiint_D 1 \, dx \, dy \, dz = \int_0^1 A(x) \, dx$$

$$= \int_0^1 \left(\iint_{T(x)} 1 \, dy \, dz \right) dx = \int_0^1 \left(\int_0^{1-x} \left(\int_0^{1-x-y} 1 \, dz \right) dy \right) dx$$

$$= \int_0^1 \left(\int_0^{1-x} \left[z \right]_0^{1-x-y} dy \right) dx$$

$$= \int_0^1 \left(\int_0^{1-x} (1-x-y) dy \right) dx = \int_0^1 \left[(1-x)y - \frac{1}{2}y^2 \right]_0^{1-x} dx$$

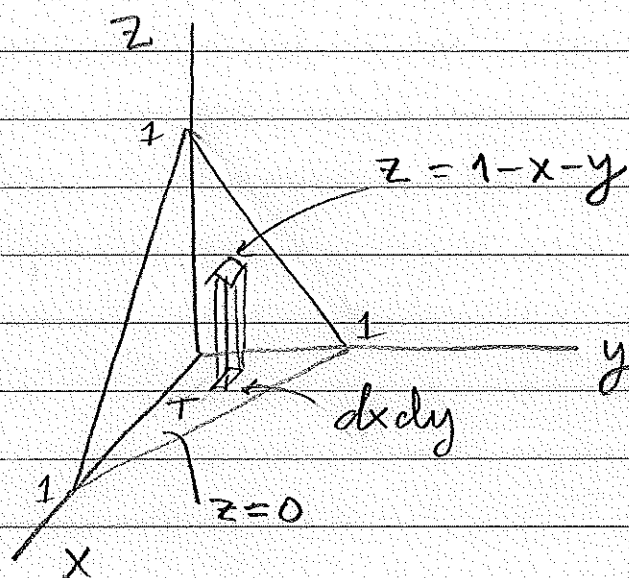
$$= \frac{1}{2} \int_0^1 (1-x)^2 dx = \frac{1}{2} \left[-\frac{(1-x)^3}{3} \right]_0^1 = \frac{1}{6}.$$

2) Mellan två grafer:

$$z = 0, (x, y) \in T$$

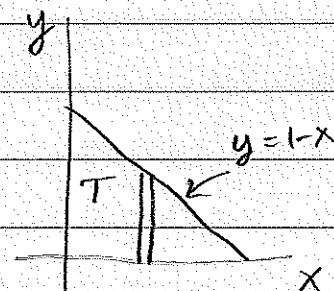
och

$$z = 1 - x - y, (x, y) \in T$$



$$V = \iint_T \left(\int_0^{1-x-y} 1 \, dz \right) dx dy =$$

$$= \iint_T (1-x-y) dx dy$$



$$= \int_0^1 \left(\int_0^{1-x} \left(\int_0^{1-x-y} 1 \, dz \right) dy \right) dx = \dots = \frac{1}{6}$$