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2016-05-03

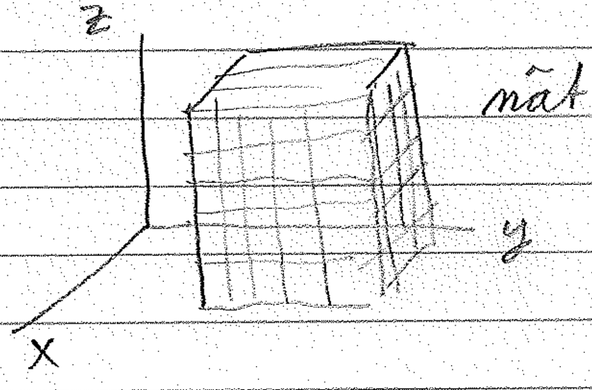
MVE255 FÖ 5.2

Idag: Trippelintegralen 14.5
Variabelbyte 14.6
Moment + masscentrum 14.7

Trippelintegralen 14.5

$$\iiint_R f(x,y,z)\,dx\,dy\,dz = \lim \sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K} f(x_{ijk})\,\Delta x_i\,\Delta y_j\,\Delta z_k$$

R rektangulärt område i $\mathbb{R}^3$ (rätblock).



Allmänt begränsat område D: integrera över stort rekt. område R så att $D \subset R$, sätt f till 0 utanför D.

Kan ibland räknas ut med upprepad integration.

Rektangulärt: $R = [a_1, a_2] \times [b_1, b_2] \times [c_1, c_2]$

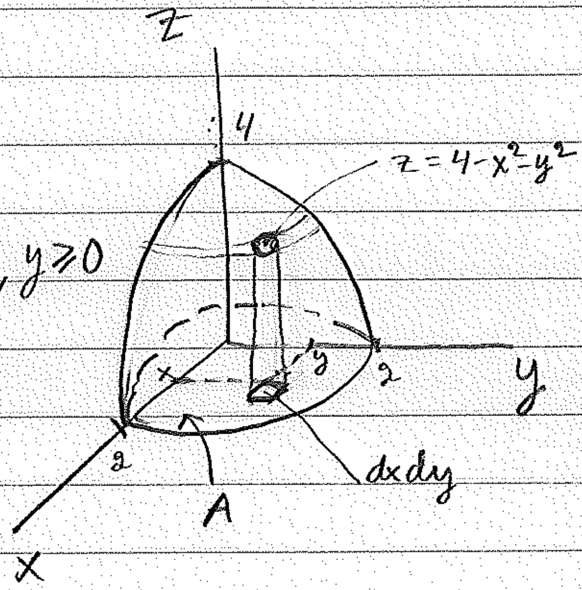
$$\iiint_R f(x,y,z\,dx\,dy\,dz = \int_{a_1}^{a_2}\left(\int_{b_1}^{b_2}\left(\int_{c_1}^{c_2} f(x,y,z)\,dz\right)dy\right)dx$$

rektangulärt, upprepad integration i vilken ordning som helst.

Allmänt område är oftast svårt.

Exempel

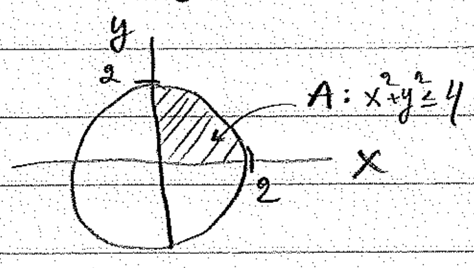
$$I = \iiint_D x \, dx\, dy\, dz$$



$D: \quad 0 \le z \le 4-x^2-y^2, \; x \ge 0, y \ge 0$

Enkelt i z, dvs
mellan två grafer
$z=0$ och $z=4-x^2-y^2$.

Bottenytan A:



$$I = \iint_A \left(\int_0^{4-x^2-y^2} x \, dz \right) dx\,dy$$

$$= \iint_A \left(x \left[z \right]_0^{4-x^2-y^2} \right) dx\,dy$$

$$= \iint_A x(4-x^2-y^2)\, dx\,dy = \{ \text{polära} \} =$$

$$= \int_0^{\pi/2} \left(\int_0^2 r\cos\theta\,(4-r^2)\, r\, dr \right) d\theta$$

$$= \int_0^{\pi/2} \cos\theta\, d\theta \int_0^2 r^2(4-r^2)\, dr = \underbrace{\left[\sin\theta \right]_0^{\pi/2}}_{=1} \underbrace{\int_0^2 \ldots\, dr}_{=\frac{64}{15}} = \frac{64}{15}$$

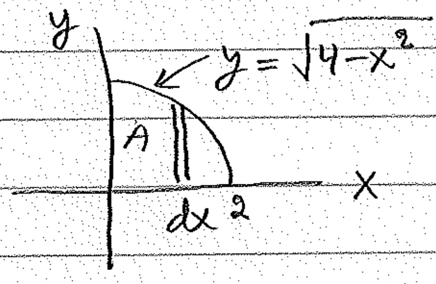
I Matlab:

```
>> f = @(x,y,z)(x.*(z <= 4-x.^2-y.^2))
>> I = integral3(f, 0,2,0,2,0,4)
```

$$\underbrace{0,2,0,2,0,4}_{[0,2]\times[0,2]\times[0,4]}$$

Se mer detaljer på sista sidan.

Alternativt:



$$I = \iint_A \left(\int_0^{4-x^2-y^2} x \, dz \right) dx\,dy =$$

$$= \int_0^2 \left(\int_0^{\sqrt{4-x^2}} \left(\int_0^{4-x^2-y^2} x \, dz \right) dy \right) dx =$$

$$= \int_0^2 \int_0^{\sqrt{4-x^2}} x(4-x^2-y^2)\, dy\, dx =$$

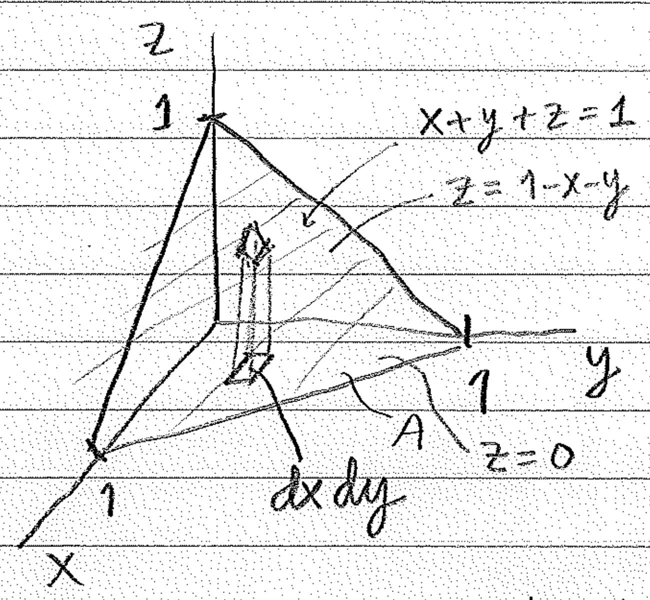
$$= \int_0^2 \left[x(4-x^2)y - x\frac{y^3}{3} \right]_0^{\sqrt{4-x^2}} dx =$$

$$= \int_0^2 \left(x(4-x^2)^{3/2} - \frac{1}{3}x(4-x^2)^{3/2} \right) dx =$$

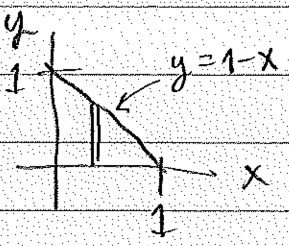
$$= \frac{1}{3}\int_0^2 2x(4-x^2)^{3/2} dx = \frac{1}{3}\left[\frac{(4-x^2)^{5/2}}{-5/2} \right]_0^2 = \frac{2}{15}4^{5/2} = \frac{64}{15}$$

Exempel Volymen av tetraeder.

T begränsas av planet $x+y+z=1$ och av koord. planen $x=0$, $y=0$ och $z=0$.



Bottenytan A:



Enkelt i z: $0 \le z \le 1-x-y$ (mellan 2 grafer)

$$V = \iiint_T 1\,dx\,dy\,dz = \iint_A \left(\int_0^{1-x-y} 1\,dz \right) dx\,dy$$

$$= \int_0^1 \left(\int_0^{1-x} \left(\int_0^{1-x-y} dz \right) dy \right) dx$$

$$= \int_0^1 \int_0^{1-x} (1-x-y)\,dy\,dx$$

$$= \int_0^1 \left[(1-x)y - \tfrac{1}{2}y^2 \right]_0^{1-x} dx$$

$$= \int_0^1 \left((1-x)^2 - \tfrac{1}{2}(1-x)^2 \right) dx$$

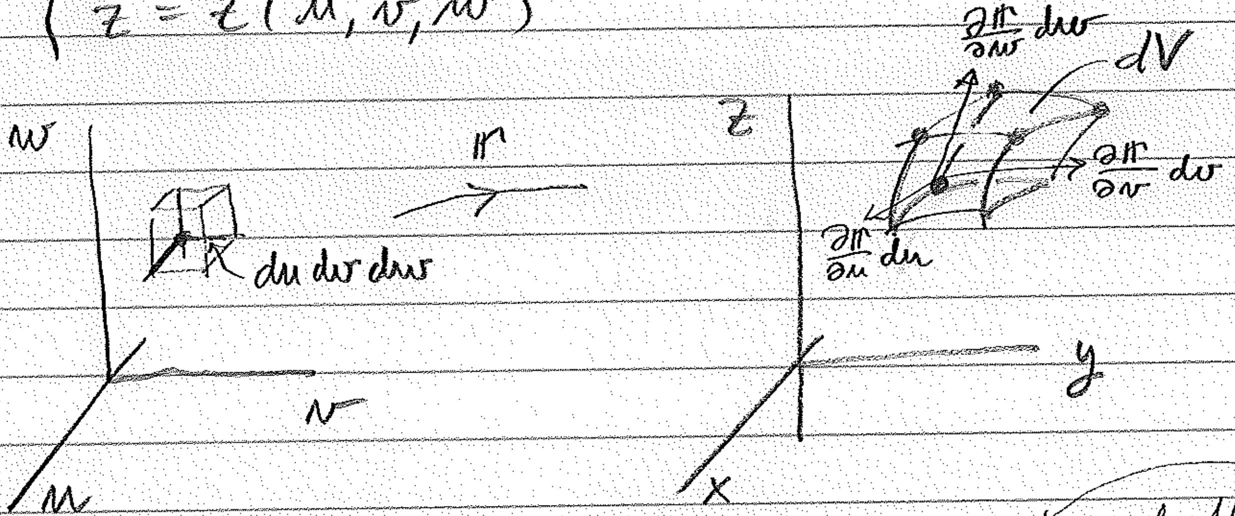
$$= \tfrac{1}{2} \int_0^1 (1-x)^2\,dx = \tfrac{1}{2}\left[-\tfrac{1}{3}(1-x)^3 \right]_0^1 = \tfrac{1}{6}.$$

I FÖ 5.1 beräknas denna med "skivmetoden".

Variabelbyte. 14.6

Nya koordinater (u, v, w).

$$\begin{cases} x = x(u,v,w) \\ y = y(u,v,w) \\ z = z(u,v,w) \end{cases} \qquad \mathbf{r} = \mathbf{r}(u,v,w)$$



dV spänns upp av tangenterna.

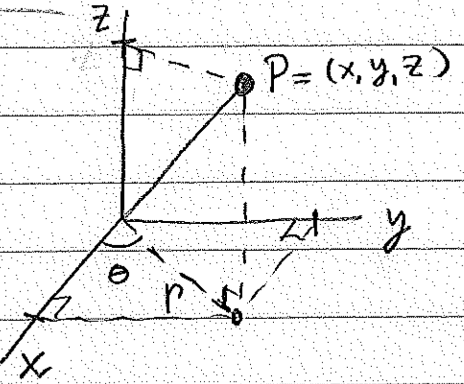
$$dV = \left| \frac{\partial \mathbf{r}}{\partial u} du \cdot \left(\frac{\partial \mathbf{r}}{\partial v} dv \times \frac{\partial \mathbf{r}}{\partial w} dw \right) \right| =$$

absolutbeloppet av trippel-produkten

$$= \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| du\, dv\, dw =$$

$$= \underbrace{\left| \det\left(\mathbf{r}'(u,v,w)\right) \right|}_{\text{absolutbeloppet av Jacobi-determinanten.}} du\, dv\, dw$$

Cylindriska koordinater (r,θ,z).



$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

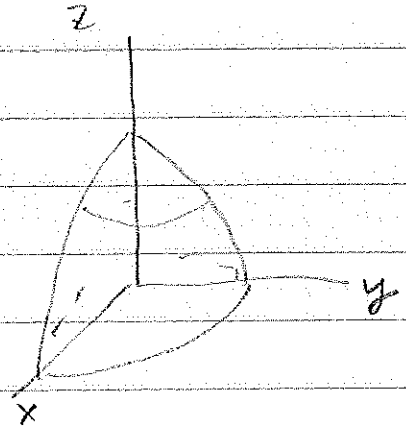
Jacobideterminanten:

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= r(\cos^2\theta + \sin^2\theta) = r$$

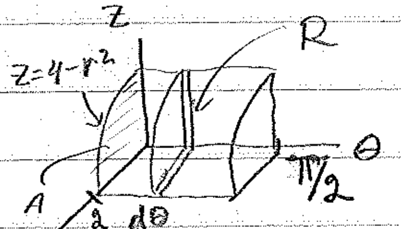
$$dV = r\,dr\,d\theta\,dz$$

Förra exemplet



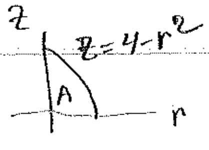
$$0 \le z \le 4 - x^2 - y^2 = 4 - r^2$$

Vi får i cyl. koord: $\begin{cases} 0 \le r \le 2 \\ 0 \le z \le 4 - r^2 \\ 0 \le \theta \le \pi/2 \end{cases}$



$$I = \iiint_D x\,dx\,dy\,dz = \iiint_R r\cos\theta\; r\,dr\,d\theta\,dz$$

$$= \int_0^{\pi/2} \left(\iint_A r\cos\theta \;\; r\,dr\,dz \right) d\theta$$

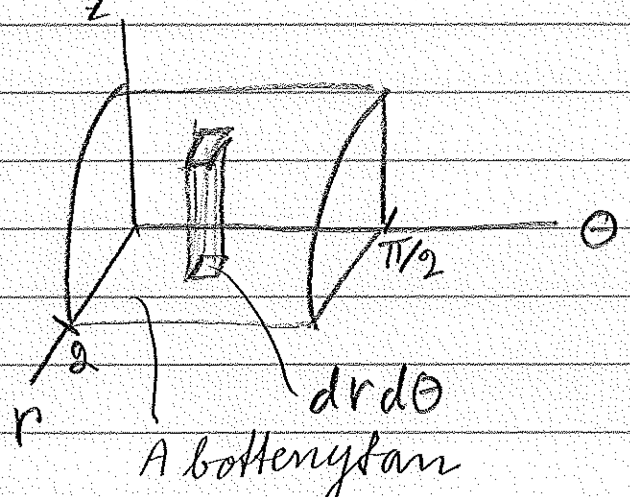


$$= \int_0^{\pi/2} \cos\theta\, d\theta \;\; \int_0^2 \left(\int_0^{4-r^2} r^2\, dz \right) dr = 1 \cdot \int_0^2 r^2(4-r^2)\, dr = $$

$$= 64/15$$

Detta var "skivmetoden".

Alternativ: R är enkelt i z



$0 \le z \le 4 - r^2, \quad (r,\theta) \in A$

$$I = \iiint_R r\cos\theta \; r\,dr\,d\theta\,dz = \iint_A \left(\int_0^{4-r^2} r^2\cos\theta\,dz \right) dr\,d\theta$$

$$= \iint_A r^2\cos\theta \, \Big[z \Big]_0^{4-r^2} \, dr\,d\theta$$

$$= \iint_A (4-r^2) r^2 \cos\theta \, dr\,d\theta = \left\{ \begin{matrix} A \text{ är} \\ \text{rektangel} \end{matrix} \right\}$$

$$= \int_0^{\pi/2} \left(\int_0^2 (4-r^2) r^2 \cos\theta \, dr \right) d\theta$$

$$= \int_0^{\pi/2} \cos\theta \, d\theta \int_0^2 (4r^2 - r^4)\, dr = 1 \cdot \frac{64}{15} = \frac{64}{15}$$

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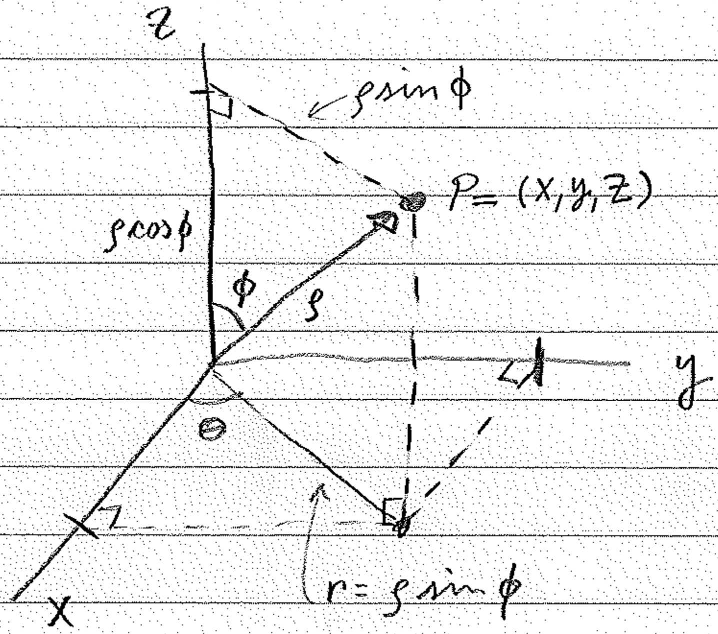
Sfäriska koordinater (ρ, ϕ, θ).

$$\begin{cases} x = \rho \sin\phi \cos\theta \\ y = \rho \sin\phi \sin\theta \\ z = \rho \cos\phi \end{cases}$$



$$\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} = \begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix} =$$

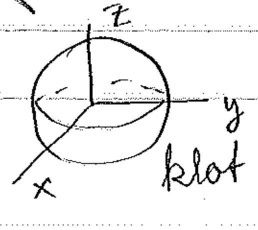
$$= \cdots = \rho^2 \sin\phi$$

obs: $\sin\phi \geq 0$ för $\phi \in [0,\pi]$

$$dV = |\rho^2 \sin\phi| \, d\rho \, d\phi \, d\theta = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

Exempel $\iiint_{B(0,1)} (x^2+y^2)\,dx\,dy\,dz = \iiint_R (\rho\sin\phi)^2 \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$

$$\begin{pmatrix} 0 \le \rho \le 1 \\ 0 \le \phi \le \pi \\ 0 \le \theta \le 2\pi \end{pmatrix} = \int_0^1 \rho^4 d\rho \int_0^\pi \sin^3\phi \, d\phi \int_0^{2\pi} d\theta = \left\{ \begin{array}{l|l} u = -\cos\phi & \phi=0,\ u=-1 \\ du = \sin\phi\, d\phi & \phi=\pi,\ u=1 \\ \sin^2\phi = 1-u^2 & \end{array} \right\}$$



$$= \frac{1}{5} \cdot \int_{-1}^{1} (1-u^2)\,du \cdot 2\pi = \frac{1}{5}\cdot\frac{4}{3}\cdot 2\pi = \frac{8\pi}{15}$$

klot med radie = 1

14.7 Endast "Moments and Centres of Mass."
sid 850 – 854

Masscentrum: $(\bar{x}, \bar{y}, \bar{z})$ där

$$\bar{x} = \frac{\iiint_R x\,\delta(x,y,z)\,dV}{\iiint_R \delta(x,y,z)\,dV}$$

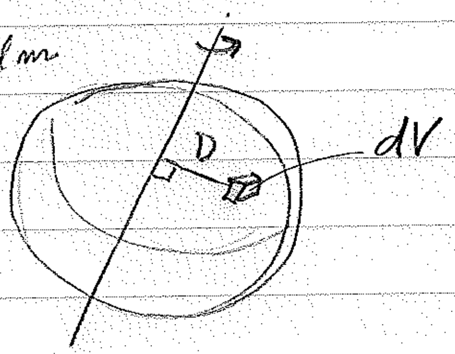
δ = masstäthet =
= densitet
$[kg/m^3]$

osv.

På vektorform: $\bar{\mathbf{r}} = \dfrac{\iiint_R \mathbf{r}\,\delta\,dV}{\iiint_R \delta\,dV} = \dfrac{\iiint_R \mathbf{r}\,dm}{\iiint_R dm}$

Tröghetsmoment m.a.p. axel:

$$I = \iiint_R D^2(x,y,z)\,\delta(x,y,z)\,dV = \iiint_R D^2\,dm$$



D = avståndet till axeln

$dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta \quad \left(\text{obs: } \sin\phi \geq 0 \right)$

Exempel. Tröghetsmomentet för enhetsklotet m.o.p. z-axeln: $\left(\begin{array}{c} \delta = \text{masstäthet} \\ \left[\frac{kg}{m^3}\right] \end{array} \right)$

$$I = \iiint_B (x^2+y^2)\delta \, dV = \delta \int_0^{2\pi}\int_0^{\pi}\int_0^1 (\rho \sin\phi)^2 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \delta \int_0^1 \rho^4 d\rho \int_0^{\pi} \sin^3\phi \, d\phi \int_0^{2\pi} d\theta = \left\{ \begin{array}{l} u = -\cos\phi \\ du = +\sin\phi \, d\phi \\ \sin^2\phi = 1-u^2 \\ \phi = 0 \Rightarrow u = -1, \phi = \pi \Rightarrow u = 1 \end{array} \right\}$$

$$= \delta \frac{1}{5} \cdot \int_{-1}^{1} (1-u^2)\, du \cdot 2\pi = \delta \frac{1}{5} \cdot \frac{4}{3} \cdot 2\pi = \frac{8\pi}{15}\delta$$

δ = masstäthet = konstant $\left[\frac{kg}{m^3}\right]$

Contents

- Example lecture 5.2. Repeated integration with integral3.
- Simpler alternative with triplequad (much less accurate)
- Integral3 cannot handle this for some reason.

Example lecture 5.2. Repeated integration with integral3.

```
format long
xmin = 0;
xmax = 2;
ymin = 0;
ymax = @(x)   sqrt(4 - x.^2);
zmin = 0;
zmax = @(x,y) 4 - x.^2 - y.^2;

f=@(x,y,z) x;
I=integral3(f,xmin,xmax,ymin,ymax,zmin,zmax)
Iexact=64/15
```

```
I =

   4.266666666666317


Iexact =

   4.266666666666667
```

Simpler alternative with triplequad (much less accurate)

```
ff=@(x,y,z) (x.*( z<= 4-x.^2-y.^2) );
II=triplequad(ff,0,2,0,2,0,4)
```

```
II =

   4.266558675930201
```

Integral3 cannot handle this for some reason.

```
ff=@(x,y,z) (x.*( z<= 4-x.^2-y.^2) );
III=integral3(ff,0,2,0,2,0,4)
```

```
Warning: Reached the maximum number of function
evaluations (10000). The result fails the global
error test.
Warning: The integration was unsuccessful.

III =
```

```
NaN
```