MVE 255 FÖ 5.3

Idag: Variabelbyte 14.4
Trippelintegralen 14.5
Variabelbyte 14.6
Masscentrum 14.7

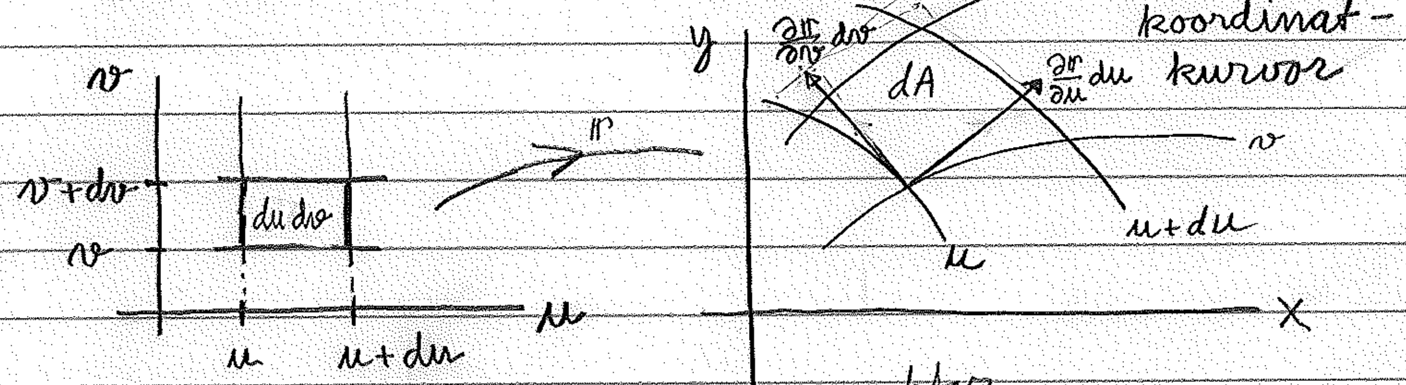
Variabelbyte

Nya koordinater (u, v).

$\mathbf{r} = \mathbf{r}(u, v)$

Transformation: $\begin{cases} x = x(u,v) \\ y = y(u,v) \end{cases}$

Invers transformation: $\begin{cases} u = u(x,y) \\ v = v(x,y) \end{cases}$



$$dA = \left| \frac{\partial \mathbf{r}}{\partial u} du \times \frac{\partial \mathbf{r}}{\partial v} dv \right|$$ längd av vektor

areaelementet

spänns upp av tangenterna $\frac{\partial \mathbf{r}}{\partial u} du$ och $\frac{\partial \mathbf{r}}{\partial v} dv$.

$$\frac{\partial \mathbf{r}}{\partial u} du \times \frac{\partial \mathbf{r}}{\partial v} dv = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial x}{\partial u} du & \frac{\partial y}{\partial u} du & 0 \\ \frac{\partial x}{\partial v} dv & \frac{\partial y}{\partial v} dv & 0 \end{vmatrix} = \left\{ \begin{matrix} \text{bryt ut} \\ du \text{ och } dv \end{matrix} \right\}$$

$$= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} du\, dv\, \mathbf{k} = \frac{\partial(x,y)}{\partial(u,v)} du\, dv\, \mathbf{k}$$

Jacobideterminanten för transformationen:

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\left(\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{bmatrix}\right) = \underbrace{\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}}_{=\text{ transponatet } \mathbb{r}'(u,v)^T \text{ av Jacobimatrisen}} \overset{(\det(A^T)=\det(A))}{=}$$

$$= \underbrace{\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}}_{= \text{ Jacobimatrisen } \mathbb{r}'(u,v)} = \det(\mathbb{r}'(u,v))$$

determinant-streck

Vi får nu

$$dA = \left|\frac{\partial \mathbb{r}}{\partial u}\,du \times \frac{\partial \mathbb{r}}{\partial v}\,dv\right| = \left|\frac{\partial(x,y)}{\partial(u,v)}\,du\,dv\,\mathbb{k}\right| =$$

längd av vektor

$$= \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

abs. belopp

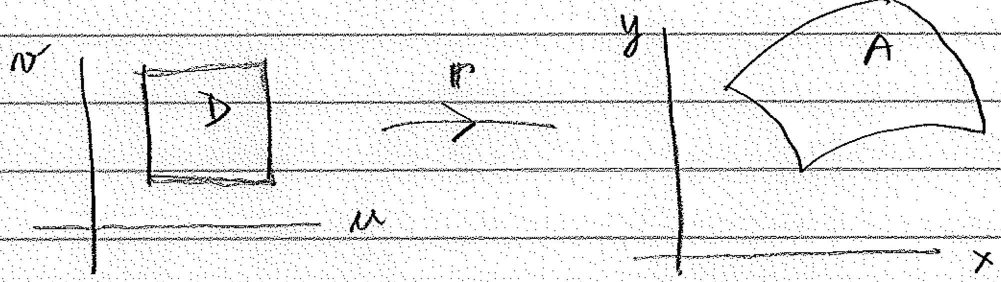
dvs

$$\boxed{\begin{aligned} dx\,dy &= \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv \\ &= |\det(\mathbb{r}'(u,v))|\,du\,dv \end{aligned}}$$

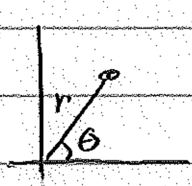
Obs: $\left|\frac{\partial(x,y)}{\partial(u,v)}\right|$ är <u>yt-skalan</u> vid transformationen.

Sats (Variabelbyte i dubbelintegral)

$$\iint_A f(x,y)\,dx\,dy = \iint_D f(\mathbf{r}(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$



Exempel Polära koord (r,θ).



$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \qquad \begin{cases} r = \sqrt{x^2+y^2} \\ \theta = \arctan\left(\frac{y}{x}\right) \end{cases}$$

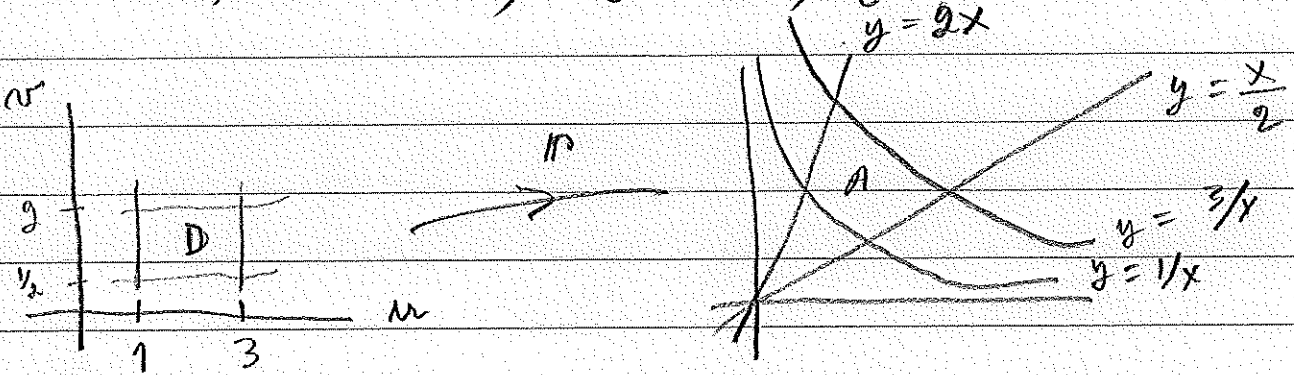
$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = r$$

$$dA = \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| dr\,d\theta = |r|\,dr\,d\theta = r\,dr\,d\theta.$$

Exempel

Beräkna $\iint_A \frac{x}{y}\,dx\,dy$ där A är området mellan kurvorna

$$y=\frac{1}{x},\quad y=\frac{3}{x},\quad y=\frac{x}{2},\quad y=2x.$$



Vi har $1\le xy\le 3$, $\frac{1}{2}\le \frac{y}{x}\le 2$. Detta bestämmer en rektangel D i uv-planet.
Välj därför $u = xy$, $v = y/x$. Detta är inversa transformationen. Lös ut x, y.

Ekv (2): $y = vx$. I ekv (1): $u = vx^2$, $x = (\pm)\sqrt{\frac{u}{v}}$

Sedan $y = v\sqrt{\frac{u}{v}} = \sqrt{uv}$.

Transformationen blir

$$\begin{cases} x = \sqrt{\frac{u}{v}} \\ y = \sqrt{uv} \end{cases}$$

Jacobideterminanten: $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} =$

$$= \begin{vmatrix} \frac{1}{2\sqrt{uv}} & -\frac{\sqrt{u}}{2\sqrt{v^3}} \\ \frac{\sqrt{v}}{2\sqrt{u}} & \frac{\sqrt{u}}{2\sqrt{v}} \end{vmatrix} = \frac{1}{2v}, \quad dA = \left|\frac{1}{2v}\right| du\,dv = \frac{1}{2v}\,du\,dv$$

Integralen blir

$$\iint_A \frac{x}{y}\,dx\,dy = \iint_D \frac{\sqrt{\frac{u}{v}}}{\sqrt{uv}}\,\frac{1}{2v}\,du\,dv =$$

$$= \iint_D \frac{1}{2v^2}\,du\,dv = \int_1^3 \left(\int_{1/2}^{2} \frac{1}{2v^2}\,dv \right) du$$

$$= \int_1^3 du \int_{1/2}^{2} \frac{1}{2v^2}\,dv = 2 \cdot \left[\frac{1}{-2v} \right]_{1/2}^{2}$$

$$= 2\left(\frac{1}{-4} - \frac{1}{-1} \right) = 2 \cdot \frac{3}{4} = \frac{3}{2}$$

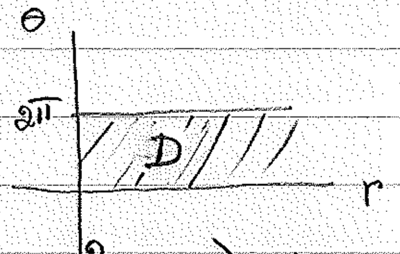
<u>Exempel</u> $\displaystyle\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$

<u>Bevis</u> Låt $I = \displaystyle\int_{-\infty}^{\infty} e^{-x^2}\,dx$. Då blir

$$I^2 = \int_{-\infty}^{\infty} e^{-x^2}\,dx \int_{-\infty}^{\infty} e^{-y^2}\,dy =$$

$$= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} e^{-x^2} e^{-y^2}\,dx \right) dy$$

$$= \iint_{\mathbb{R}^2} e^{-x^2-y^2}\,dx\,dy$$



$$\overset{\text{polära}}{=} \iint_D e^{-r^2}\, r\,dr\,d\theta = \int_0^{2\pi} \left(\int_0^{\infty} e^{-r^2}\, r\,dr \right) d\theta$$

(generaliserad integral, positiv integrand, upprepad integration är tillåten)

$$= \int_0^{2\pi} d\theta \int_0^{\infty} e^{-r^2}\, r\,dr$$

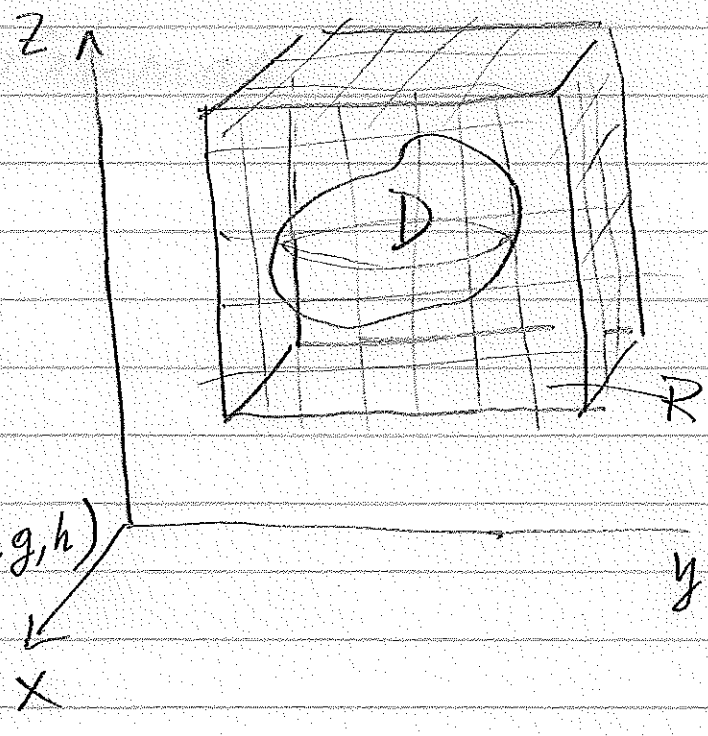
$$= 2\pi \left[-\frac{1}{2} e^{-r^2} \right]_0^{\infty} = 2\pi \cdot \frac{1}{2} = \pi$$

Trippelintegralen (14.5)

$$\iiint_D f\,dV = \iiint_D f(x,y,z)\,dx\,dy\,dz$$

definieras med partition av området och Riemann-summa.

Först över rätblock R, sedan allmänt område D

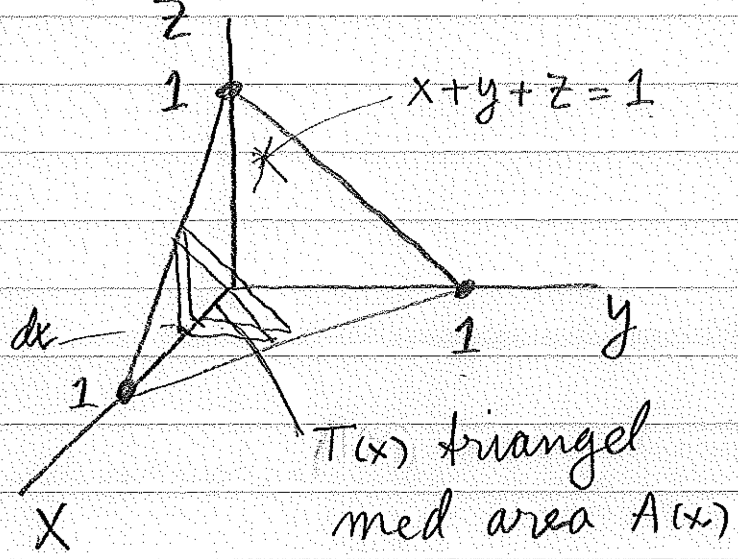


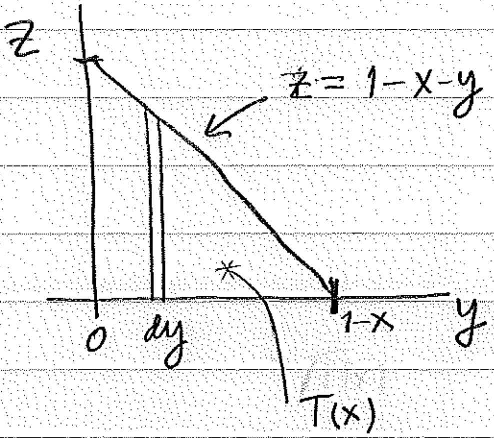
Matlab:

integral3(f, a, b, c, d, g, h)
integrerar över rätblock.

Kan ibland beräknas med upprepad integration.

Exempel Volymen av tetraeder.





Tetraedern D är enkel i alla riktningar.

1) Skivmetoden:

$$V = \iiint_D 1 \, dx\,dy\,dz = \int_0^1 A(x)\,dx$$

$$= \int_0^1 \left(\iint_{T(x)} 1 \, dy\,dz \right) dx = \int_0^1 \left(\int_0^{1-x} \left(\int_0^{1-x-y} 1 \, dz \right) dy \right) dx$$

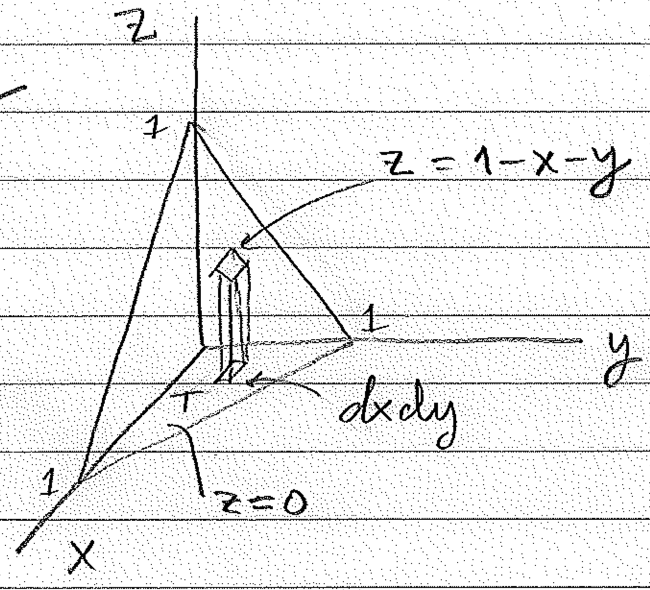
$$= \int_0^1 \left(\int_0^{1-x} \left[z \right]_0^{1-x-y} dy \right) dx$$

$$= \int_0^1 \left(\int_0^{1-x} (1-x-y)\, dy \right) dx = \int_0^1 \left[(1-x)y - \tfrac{1}{2} y^2 \right]_0^{1-x} dx$$

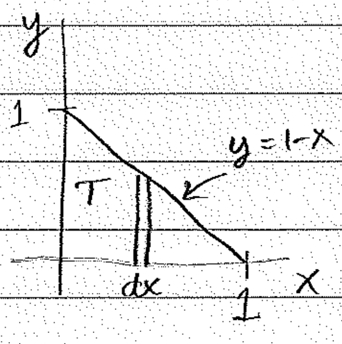
$$= \frac{1}{2} \int_0^1 (1-x)^2 \, dx = \frac{1}{2} \left[\frac{-(1-x)^3}{3} \right]_0^1 = \frac{1}{6}.$$

2) Mellan två grafer:

D är mellan
$z = 0$, $(x,y) \in T$ och
$z = 1 - x - y$, $(x,y) \in T$



$$V = \iint_T \left(\int_0^{1-x-y} 1 \, dz \right) dx\,dy =$$

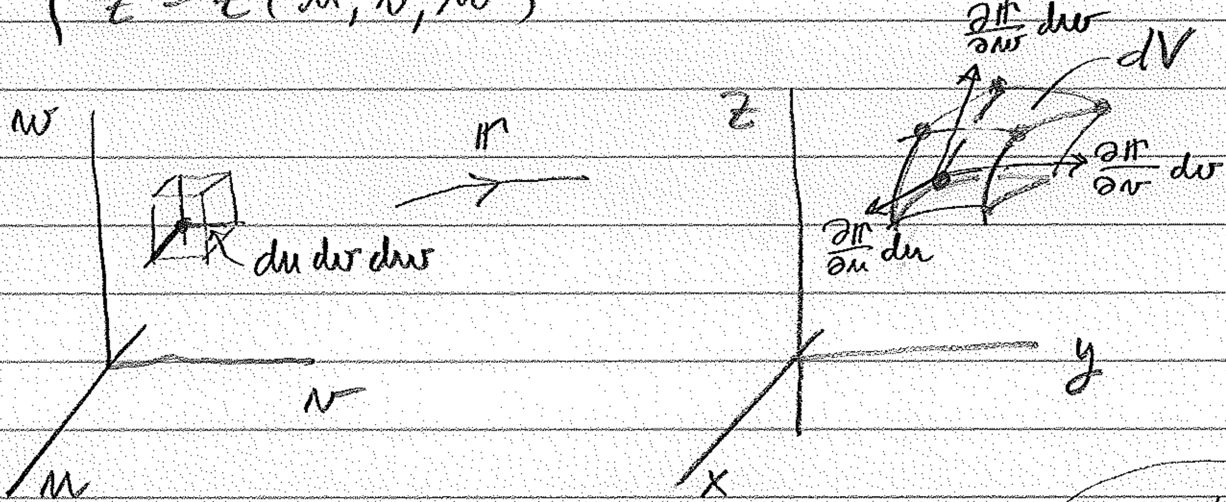


$$= \int_0^1 \left(\int_0^{1-x} \left(\int_0^{1-x-y} 1 \, dz \right) dy \right) dx = \ldots = \frac{1}{6}$$

Variabelbyte. 14.6

Nya koordinater (u, v, w).

$$\begin{cases} x = x(u, v, w) \\ y = y(u, v, w) \\ z = z(u, v, w) \end{cases} \qquad \mathbf{r} = \mathbf{r}(u, v, w)$$



dV spänns upp av tangenterna:

$$dV = \left| \frac{\partial \mathbf{r}}{\partial u} du \cdot \left(\frac{\partial \mathbf{r}}{\partial v} dv \times \frac{\partial \mathbf{r}}{\partial w} dw \right) \right| =$$

absolutbeloppet av trippelprodukten

$$= \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| du\, dv\, dw =$$

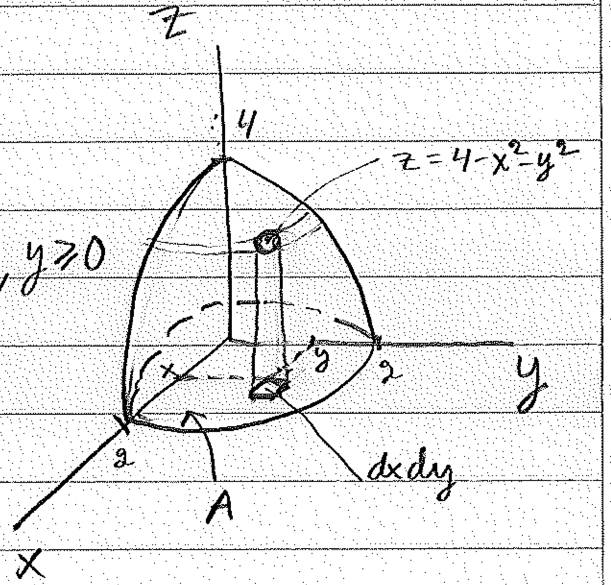
$$= \underbrace{\left| \det\left(\mathbf{r}'(u, v, w)\right) \right|}_{\text{absolutbeloppet av Jacobi-determinanten.}} du\, dv\, dw$$

Allmänt område är oftast svårt.

Exempel

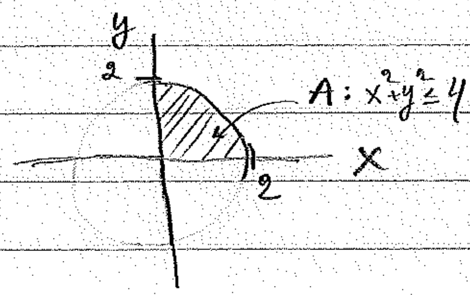
$$I = \iiint_D x \, dx\,dy\,dz$$

$D: \quad 0 \le z \le 4 - x^2 - y^2, \quad x \ge 0, \; y \ge 0$



Enkelt i z, dvs mellan två grafer $z = 0$ och $z = 4 - x^2 - y^2$.

Bottenytan A:



$$I = \iint_A \left(\int_0^{4-x^2-y^2} x \, dz \right) dx\,dy$$

$$= \iint_A \left(x \left[z \right]_0^{4-x^2-y^2} \right) dx\,dy$$

$$= \iint_A x(4 - x^2 - y^2)\, dx\,dy = \{ \text{polära} \} =$$

$$= \int_0^{\pi/2}\!\!\left(\int_0^2 r\cos\theta\,(4 - r^2)\, r\,dr \right) d\theta$$

$$= \int_0^{\pi/2} \cos\theta \, d\theta \int_0^2 r^2(4 - r^2)\,dr = \underbrace{\left[\sin\theta\right]_0^{\pi/2}}_{=1} \underbrace{\int_0^2 \cdots dr}_{=\frac{64}{15}} = \frac{64}{15}$$

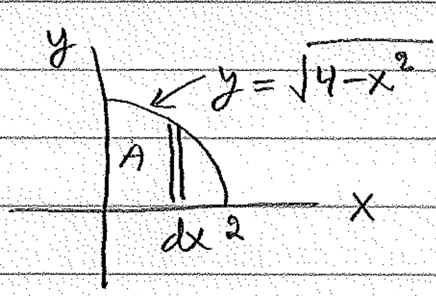
I Matlab:

```
>> f = @(x,y,z)(x.*(z <= 4-x.^2-y.^2))
```

$$\texttt{>> } I = \text{integral3}\,(f, \underbrace{0, 2, 0, 2, 0, 4}_{[0,2]\times[0,2]\times[0,4]})$$

Se mer detaljer på sista sidan.

<u>Alternativt:</u>



$$I = \iint_A \left(\int_0^{4-x^2-y^2} x \, dz \right) dx\,dy =$$

$$= \int_0^2 \left(\int_0^{\sqrt{4-x^2}} \left(\int_0^{4-x^2-y^2} x \, dz \right) dy \right) dx =$$

$$= \int_0^2 \int_0^{\sqrt{4-x^2}} x(4-x^2-y^2)\, dy\, dx =$$

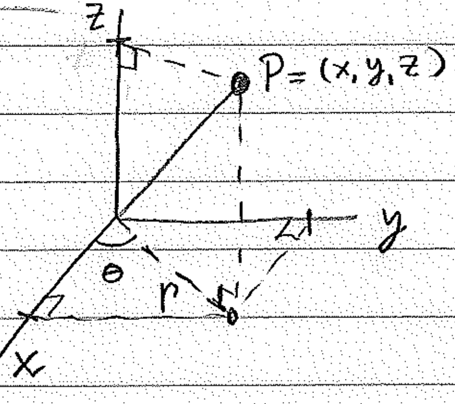
$$= \int_0^2 \left[x(4-x^2)y - x\frac{y^3}{3} \right]_0^{\sqrt{4-x^2}} dx =$$

$$= \int_0^2 \left(x(4-x^2)^{3/2} - \frac{1}{3}x(4-x^2)^{3/2} \right) dx =$$

$$= \frac{1}{3} \int_0^2 2x(4-x^2)^{3/2} dx = \frac{1}{3}\left[\frac{(4-x^2)^{5/2}}{-5/2} \right]_0^2 = \frac{2}{15} 4^{5/2} = \frac{64}{15}$$

Cylindriska koordinater (r, θ, z).

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$



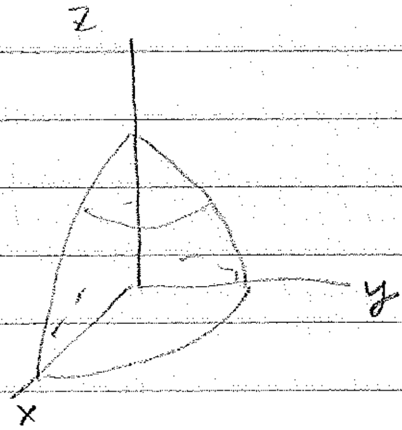
Jacobideterminanten:

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= r(\cos^2\theta + \sin^2\theta) = r$$

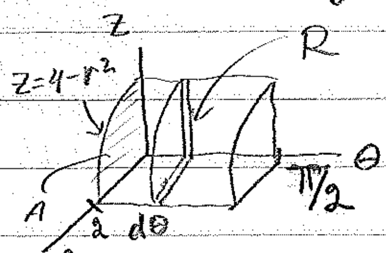
$$dV = r\,dr\,d\theta\,dz$$

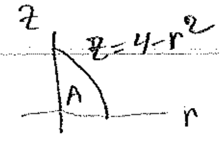
Förra exemplet



$$0 \le z \le 4 - x^2 - y^2 = 4 - r^2$$

Vi får i cyl. koord: $\begin{cases} 0 \le r \le 2 \\ 0 \le z \le 4 - r^2 \\ 0 \le \theta \le \pi/2 \end{cases}$





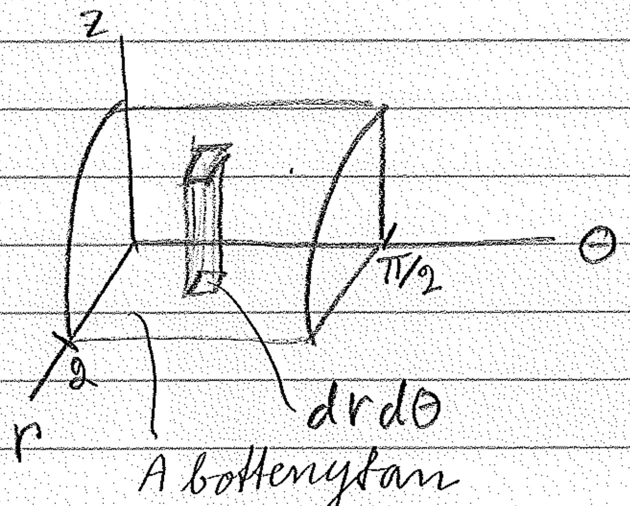
$$I = \iiint_D x\,dx\,dy\,dz = \iiint_R r\cos\theta\, r\,dr\,d\theta\,dz$$

$$= \int_0^{\pi/2} \left(\iint_A r\cos\theta \; r\,dr\,dz \right) d\theta$$

$$= \int_0^{\pi/2} \cos\theta\, d\theta \int_0^2 \left(\int_0^{4-r^2} r^2\,dz \right) dr = 1 \cdot \int_0^2 r^2(4-r^2)\,dr = 64/15$$

Detta var "skivmetoden".

Alternativ: R är enkelt i z



$0 \le z \le 4-r^2$, $(r,\theta) \in A$

$$I = \iiint_R r\cos\theta \, r \, dr \, d\theta \, dz = \iint_A \left(\int_0^{4-r^2} r^2 \cos\theta \, dz \right) dr \, d\theta$$

$$= \iint_A r^2 \cos\theta \left[z \right]_0^{4-r^2} dr \, d\theta$$

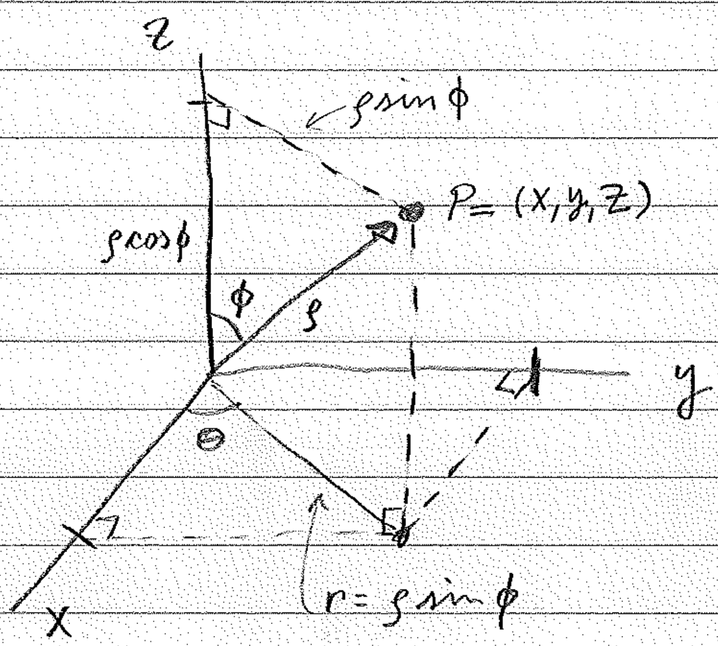
$$= \iint_A (4-r^2) r^2 \cos\theta \, dr \, d\theta = \left\{ \begin{matrix} A \text{ är} \\ \text{rektangel} \end{matrix} \right\}$$

$$= \int_0^{\pi/2} \left(\int_0^2 (4-r^2) r^2 \cos\theta \, dr \right) d\theta$$

$$= \int_0^{\pi/2} \cos\theta \, d\theta \int_0^2 (4r^2 - r^4) \, dr = 1 \cdot \frac{64}{15} = \frac{64}{15}$$

Sfäriska koordinater (ρ, ϕ, θ).

$$\begin{cases} x = \rho \sin\phi \cos\theta \\ y = \rho \sin\phi \sin\theta \\ z = \rho \cos\phi \end{cases}$$



$$\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} = \begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix} =$$

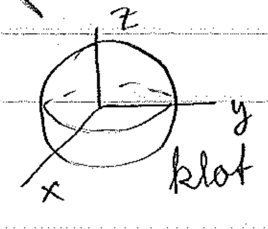
$$= \ldots = \rho^2 \sin\phi$$

obs: $\sin\phi \geq 0$ för $\phi \in [0, \pi]$

$$dV = |\rho^2 \sin\phi| \, d\rho \, d\phi \, d\theta = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

__Exempel__ $\displaystyle\iiint_{B(0,1)} (x^2+y^2)\,dx\,dy\,dz = \iiint_R (\rho\sin\phi)^2 \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$

$$\begin{pmatrix} 0 \leq \rho \leq 1 \\ 0 \leq \phi \leq \pi \\ 0 \leq \theta \leq 2\pi \end{pmatrix} = \int_0^1 \rho^4 \, d\rho \int_0^\pi \sin^3\phi \, d\phi \int_0^{2\pi} d\theta = \left\{ \begin{array}{l} u = -\cos\phi \\ du = \sin\phi\, d\phi \\ \sin^2\phi = 1-u^2 \end{array} \right| \left. \begin{array}{l} \phi = 0, u = -1 \\ \phi = \pi, u = 1 \end{array} \right\}$$



$$= \frac{1}{5} \cdot \int_{-1}^{1} (1-u^2)\,du \cdot 2\pi = \frac{1}{5} \cdot \frac{4}{3} \cdot 2\pi = \frac{8\pi}{15}$$

klot med radie = 1

14.7 Endast "Moments and Centres of Mass" sid 850–854

Masscentrum: $(\bar{x}, \bar{y}, \bar{z})$ där

$$\bar{x} = \frac{\iiint_R x\,\delta(x,y,z)\,dV}{\iiint_R \delta(x,y,z)\,dV}$$

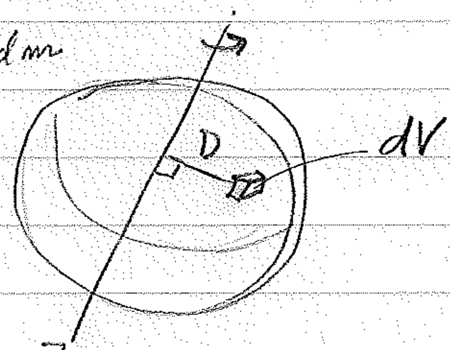
δ = masstäthet = densitet $[kg/m^3]$

osv.

På vektorform: $$\bar{\mathbb{r}} = \frac{\iiint_R \mathbb{r}\,\delta\,dV}{\iiint_R \delta\,dV} = \frac{\iiint_R \mathbb{r}\,dm}{\iiint_R dm}$$

Tröghetsmoment m.a.p. axel:

$$I = \iiint_R D^2(x,y,z)\,\delta(x,y,z)\,dV = \iiint_R D^2\,dm$$



D = avståndet till axeln

I har enheten $[kg\,m^2]$.

$dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$ (obs: $\sin\phi \geq 0$)

Exempel. Tröghetsmomentet för klot med radie R m.o.p. z-axeln: (δ = masstäthet $\left[\frac{kg}{m^3}\right]$)

$$I = \iiint_B (x^2+y^2)\delta \, dV = \delta \int_0^{2\pi}\int_0^{\pi}\int_0^{R} (\rho \sin\phi)^2 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \delta \int_0^R \rho^4 d\rho \int_0^{\pi} \sin^3\phi \, d\phi \int_0^{2\pi} d\theta = \left\{ \begin{array}{l} u = -\cos\phi \\ du = +\sin\phi \, d\phi \\ \sin^2\phi = 1-u^2 \\ \phi = 0 \Rightarrow u = -1, \phi = \pi \Rightarrow u = 1 \end{array} \right\}$$

$$= \delta \frac{R^5}{5} \cdot \int_{-1}^{1} (1-u^2)\, du \cdot 2\pi = \delta \frac{R^5}{5} \cdot \frac{4}{3} \cdot 2\pi = \frac{8\pi}{15} \delta R^5$$

δ = masstäthet = konstant $\left[\frac{kg}{m^3}\right]$

δR^5 $\left[\frac{kg}{m^3} m^5\right] = \left[kg \, m^2\right]$

Contents

- Example lecture 5.2. Repeated integration with integral3.
- Simpler alternative with triplequad (much less accurate)
- Integral3 cannot handle this for some reason.

Example lecture 5.2. Repeated integration with integral3.

```
format long
xmin = 0;
xmax = 2;
ymin = 0;
ymax = @(x)   sqrt(4 - x.^2);
zmin = 0;
zmax = @(x,y) 4 - x.^2 - y.^2;

f=@(x,y,z) x;
I=integral3(f,xmin,xmax,ymin,ymax,zmin,zmax)
Iexact=64/15
```

```
I =

   4.266666666666317


Iexact =

   4.266666666666667
```

Simpler alternative with triplequad (much less accurate)

```
ff=@(x,y,z) (x.*( z<= 4-x.^2-y.^2) );
II=triplequad(ff,0,2,0,2,0,4)
```

```
II =

   4.266558675930201
```

Integral3 cannot handle this for some reason.

```
ff=@(x,y,z) (x.*( z<= 4-x.^2-y.^2) );
III=integral3(ff,0,2,0,2,0,4)
```

```
Warning: Reached the maximum number of function
evaluations (10000). The result fails the global
error test.
Warning: The integration was unsuccessful.

III =
```

```
NaN
```