Idag: kedjeregeln 1.4
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1.4 Kedjeregeln (derivera sammansatt funktion)

En variabel:

$$g = f \circ u, \quad g(t) = f(u(t))$$

$$g'(t) = \frac{d}{dt} f(u(t)) = f'(u(t))\, u'(t)$$

yttre funktion: f, yttre derivata f'.
Inre funktion: u, inre derivata u'.

Andra beteckningar:

$$y = f(x), \quad x = u(t)$$

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

Två variabler (enkelt fall)

Partikel rör sig på ytan:

$$z = f(x,y)$$

Partikelns koordinater i x, y:

$$\begin{cases} x = u(t) \\ y = v(t) \end{cases}$$

Höjden vid tiden t:

$$z = f(u(t), v(t)):$$

Sammansatt funktion:

$$g(t) = f(u(t), v(t))$$

Yttre funk.: $f: \mathbb{R}^2 \to \mathbb{R}$

Inre funk.: $u, v: \mathbb{R} \to \mathbb{R}$

Höjdändring per tid: $\frac{dz}{dt}$ eller $g'(t)$.

Sats 1.5 (Kedjeregeln–specialfall)

Antag: $f: \mathbb{R}^2 \to \mathbb{R}$ deriverbar

$u, v: \mathbb{R} \to \mathbb{R}$ deriverbara

Då är den sammansatta funktionen $g: \mathbb{R} \to \mathbb{R}$,

$g(t) = f(u(t), v(t))$

också deriverbar med

$$g'(t) = \frac{d}{dt} f(u(t), v(t)) = f_1'(u(t), v(t))\, u'(t) + f_2'(u(t), v(t))\, v'(t)$$

Kan också skrivas

$z = f(x, y)$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Utan bevis.

Ex

$$z = x^3 - 2xy$$

$$\begin{cases} x = t^2 \\ y = \cos(t) \end{cases}$$

Kedjeregeln:

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} =$$

$$= (3x^2 - 2y)(2t) + (-2x)(-\sin(t))$$

$$= 2t(3t^2 - 2\cos(t)) + 2t^2 \sin(t)$$

På matrisform:

$$\frac{d}{dt} f(u(t), v(t)) = \left[f_1'(u(t), v(t)),\ f_2'(u(t), v(t)) \right] \begin{bmatrix} u'(t) \\ v'(t) \end{bmatrix}$$

Inre funk. med två variabler:

$$z = f(x,y), \quad \begin{cases} x = u(s,t) \\ y = v(s,t) \end{cases}$$

eller

$$g(s,t) = f(u(s,t), v(s,t))$$

Kedjeregeln med avseende på s resp. t:

$$g_1'(s,t) = \frac{\partial}{\partial s} f(u(s,t), v(s,t)) =$$

$$= f_1'(u(s,t), v(s,t))\, u_1'(s,t) + f_2'(-''-)\, v_1'(s,t)$$

$$g_2'(s,t) = \frac{\partial}{\partial t} f(u(s,t), v(s,t)) =$$

$$= f_1'(-''-)\, u_2'(s,t) + f_2'(-''-)\, v_2'(s,t)$$

eller
$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

På matrisform:

$$[g_1'(s,t), g_2'(s,t)] =$$

$$= [f_1'(u(s,t), v(s,t)), f_2'(-''-)] \begin{bmatrix} u_1'(s,t) & u_2'(s,t) \\ v_1'(s,t) & v_2'(s,t) \end{bmatrix}$$

produkt av Jacobi-matriser

<u>Sats 1.6</u> (Kedjeregeln)

$$\left(\begin{array}{ccccc} \mathbb{R}^n & \xrightarrow{g} & \mathbb{R}^m & \xrightarrow{f} & \mathbb{R}^p \\ \bar{x} & \longmapsto & \bar{y} & \longmapsto & \bar{z} \end{array} \right)$$

Antag: $g: \mathbb{R}^n \to \mathbb{R}^m$
$f: \mathbb{R}^m \to \mathbb{R}^p$

med $\bar{y} = g(\bar{x})$ och $g'(\bar{x}), f'(\bar{y})$ existar.

Då är den sammansatta funktionen

$$u = f \circ g : \mathbb{R}^n \to \mathbb{R}^p, \quad u(x) = f(g(x))$$

deriverbar i $\bar{x}$ med

$$u'(x) = f'(\bar{y})\, g'(\bar{x}).$$

Produkt av Jacobi-matriser:

$$\begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \cdots & \frac{\partial u_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial u_m}{\partial x_1} & \cdots & \frac{\partial u_m}{\partial x_n} \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial y_m} \\ \vdots & & \vdots \\ \frac{\partial f_p}{\partial x_1} & \cdots & \frac{\partial f_p}{\partial x_m} \end{bmatrix}}_{\bar{y}} \underbrace{\begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial g_m}{\partial x_1} & \cdots & \frac{\partial g_m}{\partial x_n} \end{bmatrix}}_{\bar{x}}$$

sätt in: $\bar{x}$

Utan bevis.

I en variabel: $L_f = \max_{x \in I} |f'(x)|$

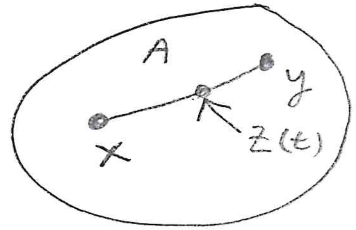
<u>Sats 1.7</u> (Deriverbar funk. är Lipschitz)

Antag: $f : \mathbb{R}^n \to \mathbb{R}^m$ deriverbar i konvex mängd $A \subseteq \mathbb{R}^n$. Då är f Lipschitz i A med

$$L_f = mn \max_{i,j} \max_{\xi \in A} \left| \frac{\partial f_i}{\partial x_j}(\xi) \right|$$

<u>Def</u> A är <u>konvex</u> om

$$x, y \in A \Rightarrow z(t) = x + t(y-x) \in A \quad \forall t \in [0,1]$$



<u>Ex</u> $I = [a,b]$

$B_R(\bar{x}) \subseteq \mathbb{R}^n$

rektangel $[a,b] \times [c,d] \subseteq \mathbb{R}^2$

Bevis: avancerat, läs själv om du vill. Bygger på

$$F(t) = f(z(t)), \quad z(t) = x + t(y-x)$$

$$F'(t) = f'(z(t))\, z'(t) = f'(z(t))(y-x)$$

och $f(y) - f(x) = F(1) - F(0) = \int_0^1 F'(t)\,dt =$

$$= \underbrace{\int_0^1 f'(z(t))\,dt}_{\left(\frac{\partial f_i}{\partial x_j}(z(t))\right)_{ij}} (x-y)$$

Tag norm!

Ex $f(x) = x_1 x_2$ på $A = [-3,3]\times[-5,5]$

I exempel 1.12: $L_f = 8$.

Sats 1.7: $m=1, n=2$

$$f'(x) = \left[\frac{\partial f}{\partial x_1}(x), \frac{\partial f}{\partial x_2}\right] = [x_2, x_1]$$

$$L_f = 1\cdot 2 \max\left(\max_{x\in A}|x_2|, \max_{x\in A}|x_1|\right)$$

$$= 2\max(5,3) = 2\cdot 5 = 10.$$

1.6 Högre ordningens derivator.

Derivera partiellt flera ggr.

$z = f(x,y)$

Rena 2:a derivator:

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial z}{\partial x} = \frac{\partial^2 f}{\partial x^2} = (f'_x)'_x(x,y) = f''_{xx}(x,y) = f''_{11}(x,y)$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 f}{\partial y^2} = f''_{yy} = f''_{22}$$

Blandade derivator:

$$\frac{\partial^2 z}{\partial x\partial y} = \frac{\partial}{\partial x}\frac{\partial z}{\partial y} = \frac{\partial^2 f}{\partial x\partial y} = (f'_y)'_x = f''_{yx} = f''_{21}$$

$$\frac{\partial^2 z}{\partial y\partial z} = \frac{\partial}{\partial y}\frac{\partial z}{\partial x} = \frac{\partial^2 f}{\partial y\partial x} = (f'_x)'_y = f''_{xy} = f''_{12}$$

Ex $f(x,y,z) = z^2 e^{x-y}$

$f'_1 = z^2 e^{x-y}$, $f''_{13} = \frac{\partial}{\partial z} z^2 e^{x-y} = 2z e^{x-y}$

$f'_3 = 2z e^{x-y}$, $f''_{31} = 2z e^{x-y}$

$f'''_{133} = 2e^{x-y}$, $f'''_{313} = 2e^{x-y}$

Obs: $f''_{13} = f''_{31}$, $f'''_{133} = f'''_{313}$

Sats 1.9 (Blandade derivator är lika.)

Om alla part. derivator är kontinuerliga så spelar det ingen roll i vilken ordning man beräknar blandade derivator.

Ex Vågekvationen.

$$w(x,t) = f(x-ct) + g(x+ct)$$

med deriverbara $f, g : \mathbb{R} \to \mathbb{R}$

och $c > 0$.

Uppfyller vågekvationen

$$w''_{tt} = c^2 w''_{xx}$$

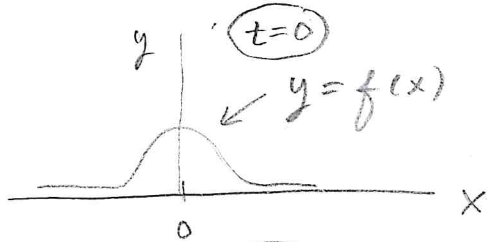
$$w'_x = f'(x-ct) + g'(x+ct)$$

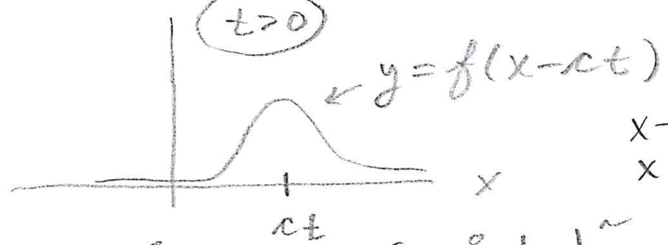
$$w''_{xx} = f''(x-ct) + g''(x+ct)$$

$$w'_t = -c f'(x-ct) + c g'(x+ct)$$

$$w''_{tt} = c^2 f''(x-ct) + c^2 g''(x+ct)$$

$$\Rightarrow w''_{tt} = c^2 w''_{xx}$$





$x - ct = 0$
$x = ct$

$f(x-ct)$: våg rör sig åt höger

$g(x+ct)$: våg åt vänster