Problem 2.1(

$$y=3-e^{2x}+2e^{x} \iff \left/ \begin{matrix}\text{variabelbyte}\\ z=e^{x}\end{matrix} \right/ \iff y=3-z^{2}+2z$$

$$\iff 0=z^{2}-2z+y-3$$

$$\iff z_{1,2}=1\pm\sqrt{1-(y-3)}=1\pm\sqrt{4-y}$$

(a) om $x\geq 0$, så $z=e^{x}\geq e^{0}=1$, alltså $z\geq 1$

Av de två rötterna ovan är det bara $1+\sqrt{4-y}$ som är större än (el. lika med) 1.

$\therefore \quad z=1+\sqrt{4-y}$, där $4-y\geq 0$, alltså $y\leq 4$

$$z=e^{x} \Rightarrow e^{x}=1+\sqrt{4-y} \iff \boxed{\begin{matrix}x=\ln\left(1+\sqrt{4-y}\right)\\ \text{där } y\leq 4\end{matrix}}$$

$\therefore \quad f^{-1}(x)=\ln\left(1+\sqrt{4-x}\right), \quad \mathcal{D}_{f^{-1}}=(-\infty,4]$

(b) om $x\leq 0$, så $0<z=e^{x}\leq e^{0}=1$, alltså $0<z\leq 1$

Av de två rötterna $z_{1,2}$ ovan är det bara $1-\sqrt{4-y}$ som kan ligga mellan 0 & 1.

$\therefore \quad z=1-\sqrt{4-y}$, där $4-y\geq 0$ (alltså $y\leq 4$)

& $1-\sqrt{4-y}>0$

$$1-\sqrt{4-y}>0 \iff 1>\sqrt{4-y} \iff 1^{2}>4-y\geq 0$$

$$\iff -1<y-4\leq 0$$

$$\iff 3<y\leq 4$$

$$z=e^{x} \Rightarrow e^{x}=1-\sqrt{4-y} \iff \boxed{\begin{matrix}x=\ln\left(1-\sqrt{4-y}\right)\\ \text{där } 3<y\leq 4\end{matrix}}$$

$\therefore \quad f^{-1}(x)=\ln\left(1-\sqrt{4-x}\right), \quad \mathcal{D}_{f^{-1}}=(3,4]$

Problem 2.2

$\sinh^{-1}$: Man ska lösa ut x ur ekvationen

$$y = \frac{e^{x}-e^{-x}}{2} \quad /\text{variabelbyte } z = e^{x}/$$

$$y = \frac{z - \frac{1}{z}}{2} \iff 2y = z - \frac{1}{z} \iff 2yz = z^2 - 1$$

$$\iff z^2 - 2yz - 1 = 0$$

$$\iff z_{1,2} = y \pm \sqrt{y^2+1} \quad (\text{p,q-formeln})$$

Den negativa lösningen ska kastas eftersom $z = e^{x}$, vilket alltid är positivt.

$$e^{x} = y + \sqrt{y^2+1} \iff x = \ln\left(y + \sqrt{y^2+1}\right)$$

$$\therefore \sinh^{-1}(y) = \ln\left(y+\sqrt{y^2+1}\right) \text{ där } y \in \mathbb{R}.$$

$\cosh^{-1}$: Lös ut x ur ekvationen $y = \frac{e^{x}+e^{-x}}{2}$. Sätt $z = e^{x}$

$$y = \frac{z + \frac{1}{z}}{2} \iff 2yz = z^2 + 1 \iff z^2 - 2yz + 1 = 0$$

$$\iff z_{1,2} = \underbrace{y \pm \sqrt{y^2-1}}_{\text{detta är positivt då } y \geq 1} \quad \ldots \text{ detta är definierat då } |y| \geq 1$$

$$x_1 = \ln\left(y + \sqrt{y^2-1}\right)$$

$$x_2 = \ln\left(y - \sqrt{y^2-1}\right) = -\ln\left(y + \sqrt{y^2-1}\right)$$

$$\therefore \cosh^{-1}(y) = \ln\left(y + \sqrt{y^2-1}\right) \qquad \cosh_{*}^{-1}(y) = \ln\left(y - \sqrt{y^2-1}\right)$$

$$y \geq 1$$

$\tanh^{-1}$: Lös ut x ur ekvationen $y = \frac{e^{x}-e^{-x}}{e^{x}+e^{-x}} = \frac{e^{2x}-1}{e^{2x}+1}$. Sätt $z = e^{2x}$

$$y = \frac{z-1}{z+1} \iff yz + y = z - 1 \iff 1 + y = z \cdot (1-y)$$

$$\iff z = \frac{1+y}{1-y} \iff e^{2x} = \frac{1+y}{1-y} \iff 2x = \ln\left(\frac{1+y}{1-y}\right)$$

$$\therefore \tanh^{-1}(y) = \frac{1}{2}\ln\left(\frac{1+y}{1-y}\right) \text{ där } \frac{1+y}{1-y} > 0, \text{ d.v.s. } y \in (-1,1)$$

Problem 2.31

(a) $$\sinh x \cosh y + \sinh y \cosh x = \frac{e^x - e^{-x}}{2} \cdot \frac{e^y + e^{-y}}{2} + \frac{e^x + e^{-x}}{2} \cdot \frac{e^y - e^{-y}}{2}$$

$$= \frac{1}{4} \cdot \left(e^{x+y} + \cancel{e^{x-y}} - \cancel{e^{-x+y}} - e^{-x-y}\right) + \frac{1}{4}\left(e^{x+y} - \cancel{e^{x-y}} + \cancel{e^{-x+y}} - e^{-x-y}\right)$$

$$= \frac{1}{4} \cdot \left(2e^{x+y} - 2e^{-x-y}\right) = \frac{e^{x+y} - e^{-(x+y)}}{2} = \sinh(x+y)$$

$\square$

(b) $$\cosh x \cosh y + \sinh x \sinh y = \frac{e^x + e^{-x}}{2} \cdot \frac{e^y + e^{-y}}{2} + \frac{e^x - e^{-x}}{2} \cdot \frac{e^y - e^{-y}}{2}$$

$$= \frac{1}{4}\left(e^{x+y} + \cancel{e^{x-y}} + \cancel{e^{-x+y}} + e^{-x-y}\right) + \frac{1}{4} \cdot \left(e^{x+y} - \cancel{e^{x-y}} - \cancel{e^{-x+y}} + e^{-x-y}\right)$$

$$= \frac{1}{4} \cdot \left(2e^{x+y} + 2e^{-x-y}\right) = \frac{e^{x+y} + e^{-(x+y)}}{2} = \cosh(x+y)$$

$\square$

(c) $$\tanh(x+y) = \frac{\sinh(x+y)}{\cosh(x+y)} = \frac{\sinh x \cosh y + \sinh y \cosh x}{\cosh x \cosh y + \sinh x \sinh y}$$

= / bryt ut $\cosh x \cosh y$ /

$$= \frac{\cancel{\cosh x} \cdot \cancel{\cosh y} \cdot \left(\frac{\sinh x}{\cosh x} \cdot 1 + \frac{\sinh y}{\cosh y} \cdot 1\right)}{\cancel{\cosh x} \cdot \cancel{\cosh y} \left(1 \cdot 1 + \frac{\sinh x}{\cosh x} \cdot \frac{\sinh y}{\cosh y}\right)}$$

$$= \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$$

$\square$

(d) $$\cosh^2 x - \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2$$

$$= \frac{\cancel{e^{2x}} + 2e^x \cdot e^{-x} + \cancel{e^{-2x}}}{4} - \frac{\cancel{e^{2x}} - 2 \cdot e^x \cdot e^{-x} + \cancel{e^{-2x}}}{4} = \frac{4e^x e^{-x}}{4}$$

$= e^0 = 1$

$$= \frac{4}{4} \cdot e^0 = 1 \cdot 1 = 1$$

$\square$

Problem 2.4

(i) $\ln(x+y) = \ln x + \ln y$

(a) $x = y = 1 \rightsquigarrow$ $\left.\begin{array}{l} VL = \ln 2 \approx 0{,}693 \\ HL = \ln 1 + \ln 1 = 0 + 0 = 0 \end{array}\right\} VL \neq HL$

(b) $x = y = 2 \rightsquigarrow$ $\left.\begin{array}{l} VL = \ln 4 = \ln 2^2 = 2\ln 2 \approx 1{,}386 \\ HL = \ln 2 + \ln 2 = 2\ln 2 \end{array}\right\} VL = HL$

(c) Förutsatt att $x > 0$ & $y > 0$:

$$\ln(x+y) = \ln x + \ln y \iff \ln(x+y) = \ln(x \cdot y)$$

$$\iff x + y = x \cdot y \iff x = x \cdot y - y \iff x = (x-1) \cdot y$$

$$\iff \boxed{y = \frac{x}{x-1}}$$

$y > 0$ uppfylls om $\frac{x}{x-1} > 0$, vilket uppfylls om $x > 1$ el. $x < 0$. Vi har dock utgått från att $x > 0$ och så kan det inte bli $x < 0$.

$\therefore$ $\underline{y = \frac{x}{x-1} \text{ där } x > 1}$

(ii) $\ln(x-y) = \ln x - \ln y$

(a) $x = 2,\ y = 1 \rightsquigarrow$ $\left.\begin{array}{l} VL = \ln(2-1) = \ln 1 = 0 \\ HL = \ln 2 - \ln 1 = \ln 2 - 0 = \ln 2 \approx 0{,}693 \end{array}\right\} VL \neq HL$

(b) $x = 4,\ y = 2 \rightsquigarrow$ $\left.\begin{array}{l} VL = \ln(4-2) = \ln 2 \\ HL = \ln 4 - \ln 2 = 2\ln 2 - \ln 2 = \ln 2 \end{array}\right\} VL = HL$

(c) Förutsatt att $x > 0$, $y > 0$ & $x - y > 0$:

$$\ln(x-y) = \ln x - \ln y \iff \ln(x-y) = \ln\frac{x}{y} \iff x - y = \frac{x}{y}$$

$$\iff xy - y^2 = x \iff y^2 - xy + x = 0$$

$$\iff y_{1,2} = \frac{x}{2} \pm \frac{\sqrt{x^2-4x}}{2}$$ ... det krävs att $x^2 - 4x \geq 0$, så $x \geq 4$

Det är lätt att se att $y_{1,2} > 0$ & $x > y_{1,2}$ då.

$\therefore$ $y = \frac{x + \sqrt{x^2-4x}}{2}$ eller $y = \frac{x - \sqrt{x^2-4x}}{2}$, där $x \geq 4$.

(iii) $e^{x+y} = e^x + e^y$

(a) $x=y=0$: $VL = e^{0+0} = e^0 = 1$, $HL = e^0 + e^0 = 1+1 = 2$ $\Rightarrow VL \neq HL$

(b) $x=y=\ln 2$: $VL = e^{\ln 2+\ln 2} = e^{\ln 2} \cdot e^{\ln 2} = 2 \cdot 2 = 4$, $HL = e^{\ln 2} + e^{\ln 2} = 2+2 = 4$ $\Rightarrow VL = HL$

(c) $e^{x+y} = e^x + e^y \iff e^x \cdot e^y - e^y = e^x \iff e^y \cdot (e^x - 1) = e^x$

$$\iff e^y = \frac{e^x}{e^x-1} \iff y = \ln \frac{e^x}{e^x-1} \iff y = x - \ln(e^x - 1)$$

det kräos att $e^x - 1 > 0 \iff e^x > 1 \iff x > 0$

$\therefore$ $y = x - \ln(e^x - 1)$, där $x > 0$

(iv) $e^{x-y} = e^x - e^y$

(a) $x=y=0$: $VL = e^{0-0} = e^0 = 1$, $HL = e^0 - e^0 = 1-1 = 0$ $\Rightarrow VL \neq HL$

(b) $x = \ln 4$, $y = \ln 2$: $VL = e^{\ln 4 - \ln 2} = e^{\ln 4} \div e^{\ln 2} = 4 \div 2 = 2$, $HL = e^{\ln 4} - e^{\ln 2} = 4 - 2 = 2$ $\Rightarrow VL = HL$

(c) $$e^{x-y} = e^x - e^y \iff \frac{e^x}{e^y} = e^x - e^y \iff e^x = e^x \cdot e^y - (e^y)^2$$

$$\iff /\text{variabelbyte } z = e^y/ \iff e^x = e^x \cdot z - z^2$$

$$\iff 0 = z^2 - e^x \cdot z + e^x \iff z_{1,2} = \frac{e^x \pm \sqrt{e^{2x} - 4e^x}}{2}$$

Det kräos att $e^{2x} - 4e^x \geq 0 \iff e^x \cdot (e^x - 4) \geq 0$

$\iff e^x \geq 4 \iff x \geq \ln 4$

$z = e^y \rightarrow y = \ln z$

$\therefore$ $y_{1,2} = \ln \frac{e^x \pm \sqrt{e^{2x} - 4e^x}}{2}$, där $x \geq \ln 4$

Problem 2.5

(i) $\ln x^p = (\ln x)^p$

(a) $x=e,\ p=2$: $\left.\begin{aligned} VL &= \ln e^2 = 2 \\ HL &= (\ln e)^2 = 1^2 = 1 \end{aligned}\right\}\ VL \neq HL$

(b) $p=1,\ x>0$: $\left.\begin{aligned} VL &= \ln x \\ HL &= \ln x \end{aligned}\right\}\ VL = HL$

(c) Förutsatt att $x>0$ ($x>1$ ifall p ej är heltal):

$$\ln x^p = (\ln x)^p \iff p \cdot \ln x = (\ln x)^p \iff 0 = (\ln x)^p - p\ln x$$

$(p \neq 0)$

$$\iff 0 = \ln x \cdot \left((\ln x)^{p-1} - p\right) \iff \ln x = 0 \text{ eller } (\ln x)^{p-1} = p$$

$\ln x = 0 \iff \boxed{x=1}$ ... det krävs att $p>0$, ty annars är HL odefinierat

$(\ln x)^{p-1} = p \iff \boxed{p=1}$ eller $\ln x = \sqrt[p-1]{p}$

$\boxed{x = e^{\sqrt[p-1]{p}}}$

$\therefore$ $p>0$ o $x=1$ el. $p=1$ o $x>0$

eller $p \neq 1$, $p \neq 0$ o $x = e^{\sqrt[p-1]{p}}$

ifall $p<0$, så måste p vara jämnt heltal

(ii) $(e^x)^p = e^{x^p}$

(a) $x=1,\ p=2$: $\left.\begin{aligned} VL &= (e^1)^2 = e^2 \approx 7{,}389 \\ HL &= e^{1^2} = e^1 = e \approx 2{,}718 \end{aligned}\right\} \Rightarrow VL \neq HL$

(b) $x=p=1$: $\left.\begin{aligned} VL &= (e^1)^1 = e^1 = e \\ HL &= e^{1^1} = e^1 = e \end{aligned}\right\}\ VL = HL$

(c) Förutsatt att $x>0$ ifall p ej är heltal.

$$(e^x)^p = e^{x^p} \iff e^{xp} = e^{x^p} \iff xp = x^p \iff \begin{cases} p=1,\ x \in \mathbb{R} \\ x=0,\ p>0 \\ x = \sqrt[p-1]{p},\ p \neq 0,1 \end{cases}$$