MVE425c, Lösningar omtentamen 12/4-17

1 (a) $D(x\cdot 2^x) = 1\cdot 2^x + x\cdot D(2^x) =$

$= 2^x + x\cdot D(e^{x\ln(2)}) = 2^x + x\cdot e^{x\ln(2)}\cdot \ln(2) =$

$= 2^x + x\cdot 2^x \ln(2) = 2^x(1 + x\ln(2))$

(b) $D\left(\frac{\ln(x^2-1)}{x}\right) = \frac{D(\ln(x^2-1))\cdot x - D(x)\cdot \ln(x^2-1)}{x^2} =$

$= \frac{\frac{2x}{x^2-1}\cdot x - 1\cdot \ln(x^2-1)}{x^2} = \frac{2x^2 - (x^2-1)\ln(x^2-1)}{x^2(x^2-1)}$

(c) $D\left(\frac{1}{\arctan(x^2)}\right) = D\left((\arctan(x^2))^{-1}\right) =$

$= -(\arctan(x^2))^{-2}\cdot D\arctan(x^2) =$

$= -\frac{1}{(\arctan(x^2))^2}\cdot\frac{1}{1+(x^2)^2}\cdot D(x^2) =$

$= -\frac{2x}{(1+x^4)(\arctan(x^2))^2}$

2. Tänk $y = y(x)$ och derivera båda leden m.a.p. x

$$\frac{3y^2 \cdot y' \cdot x - y^3}{x^2} + 2xy + x^2 \cdot y' = 0$$

Stoppa in $(-2, -2)$:

$$\frac{3 \cdot (-2)^2 \cdot y' \cdot (-2) - (-2)^3}{(-2)^2} + 2 \cdot (-2) \cdot (-2) + (-2)^2 \cdot y' = 0$$

$$\Leftrightarrow \frac{-24y' + 8}{4} + 8 + 4y' = 0$$

$$\Leftrightarrow -6y' + 2 + 8 + 4y' = 0$$

$$\Leftrightarrow 2y' = 10 \Leftrightarrow y'\Big|_{(-2,-2)} = 5 = k_T$$

Enpunktsformeln: $y - (-2) = 5(x - (-2))$

$$\Leftrightarrow y = 5x + 10 - 2 \Leftrightarrow \underline{\underline{y = 5x + 8}}$$

3. Steg 1: $D_f = (0, \infty)$

Steg 2: I. Lodräta asymptoter:

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\left(\frac{1}{2x} - \frac{2}{\sqrt{x}}\cdot\frac{2\sqrt{x}}{2\sqrt{x}}\right) = \lim_{x\to 0^+}\frac{1-4\sqrt{x}}{2x} = \infty$$

II. Vågräta asymptoter:

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\frac{1-4\sqrt{x}}{2x} = \lim_{x\to\infty}\frac{\cancel{\sqrt{x}}\left(\frac{1}{\sqrt{x}}-4\right)}{\cancel{\sqrt{x}}\cdot 2\sqrt{x}} = 0$$

$\therefore$ $x=0$ lodrät asymptot

$y=0$ vågrät asymptot

Inga sneda asymptoter

Steg 3: $f'(x) = -\frac{1}{2x^2} + \frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{x^{3/2}}$

Kritisk punkt

$$f'(x)=0 \iff \frac{1}{x^{3/2}} = \frac{1}{2x^2} \iff x^{1/2} = \frac{1}{2} \iff x = \frac{1}{4}$$

Steg 4:

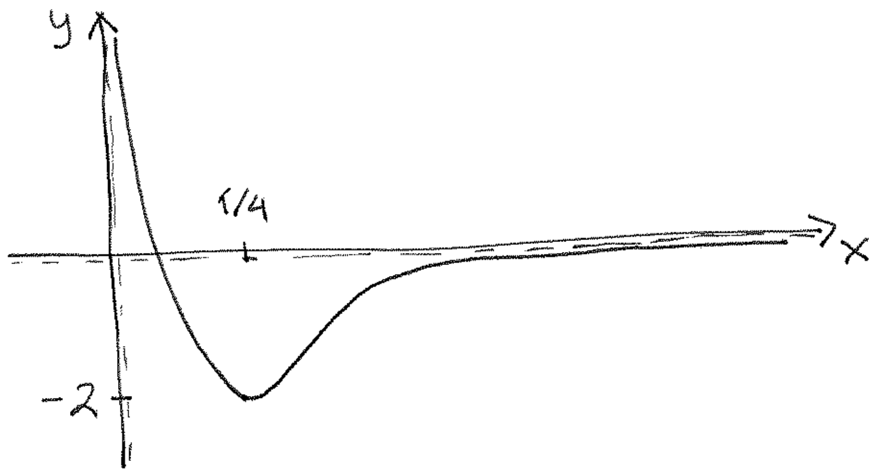
	0^+		$1/4$		∞
f'		$---$	0	$+++$	
f	∞	$\searrow$	-2	$\nearrow$	0

$$f\left(\frac{1}{4}\right) = \frac{1}{2\cdot\frac{1}{4}} - \frac{2}{\sqrt{\frac{1}{4}}} = 2 - 2\cdot 2 = -2$$

$$f'\left(\frac{1}{100}\right) = -\frac{10^4}{2} + 10^3 < 0$$

$$f'(100) = -\frac{1}{2\cdot 10^4} + \frac{1}{10^3} = \frac{1}{10^3}\left(1 - \frac{1}{20}\right) > 0$$

Steg 5:



4. $D_f = (-1,0) \cup (0,\infty)$

$$\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} \frac{1}{x} + 2\ln\underbrace{(x+1)}_{\to 0^+} = -\infty$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{1}{x} + 2\ln(x+1) = -\infty$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{1}{x} + 2\ln(x+1) = \infty$$

$$\lim_{x\to \infty} f(x) = \lim_{x\to \infty} \frac{1}{x} + 2\ln(x+1) = \infty$$

$$f'(x) = -\frac{1}{x^2} + \frac{2}{1+x}$$

$$f'(x) = 0 \iff \frac{2}{1+x} = \frac{1}{x^2} \iff 2x^2 = x+1$$

$$\iff 2x^2 - x - 1 = 0 \iff x^2 - \frac{1}{2}x - \frac{1}{2} = 0$$

$$\Rightarrow x = \frac{1}{4} \pm \sqrt{\frac{1}{16} + \frac{1\cdot 8}{2\cdot 8}} = \frac{1}{4} \pm \frac{3}{4}$$

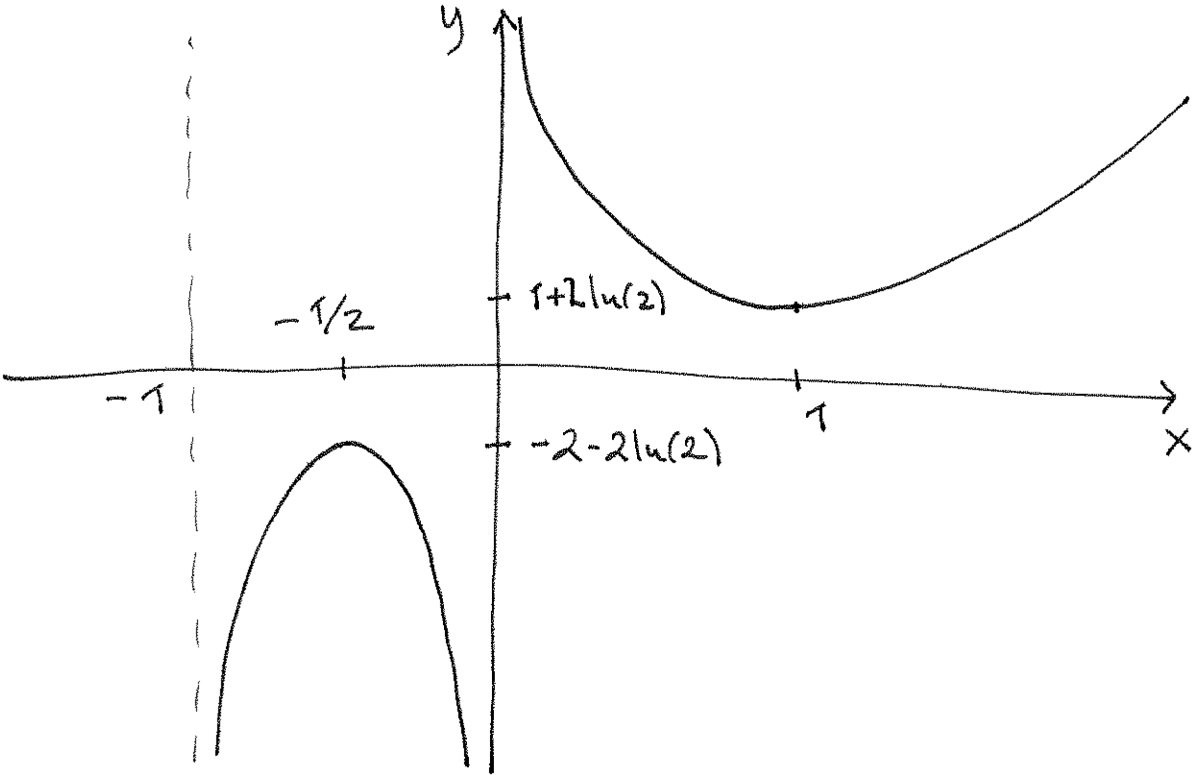
$$\therefore x_1 = 1, \quad x_2 = -\frac{1}{2}$$

$$f(1) = \frac{1}{1} + 2\ln(1+1) = 1 + 2\ln(2) > 0$$

$$f\left(-\frac{1}{2}\right) = -\frac{1}{\frac{1}{2}} + 2\ln\left(-\frac{1}{2}+1\right) = -2 + 2\ln(2^{-1}) =$$

$$= -2 - 2\ln(2) < 0$$

	-1^+		$-1/2$		0^-	0^+		1		∞
f'		$+$	0	$-$			$-$	0	$+$	
f	$-\infty$	$\nearrow$	$\uparrow$ $-2-2\ln(2)$	$\searrow$	$-\infty$	$+\infty$	$\searrow$	$\uparrow$ $1+2\ln(2)$	$\nearrow$	∞



$\therefore \quad V_f = (-\infty, -2-2\ln(2)] \cup [1+2\ln(2), \infty)$

5. $$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} =$$

$$= \lim_{h\to 0} \frac{\frac{1}{\sqrt{1+x+h}} - \frac{1}{\sqrt{1+x}}}{h} =$$

$$= \lim_{h\to 0} \frac{\sqrt{1+x} - \sqrt{1+x+h}}{h\sqrt{1+x}\sqrt{1+x+h}} =$$

$$= \lim_{h\to 0} \frac{(\sqrt{1+x} - \sqrt{1+x+h})(\sqrt{1+x} + \sqrt{1+x+h})}{h\sqrt{1+x}\sqrt{1+x+h}(\sqrt{1+x}+\sqrt{1+x+h})} =$$

$$= \lim_{h\to 0} \frac{1+x-(1+x+h)}{h\sqrt{1+x}\sqrt{1+x+h}(\sqrt{1+x}+\sqrt{1+x+h})} =$$

$$= \lim_{h\to 0} \frac{-\cancel{h}1}{\cancel{h}\sqrt{1+x}\sqrt{1+x+h}(\sqrt{1+x}+\sqrt{1+x+h})} =$$

$$= -\frac{1}{\sqrt{1+x}\sqrt{1+x}(\sqrt{1+x}+\sqrt{1+x})} =$$

$$= -\frac{1}{(1+x)\cdot 2\sqrt{1+x}} = -\frac{1}{2(1+x)^{3/2}}$$

6. $D_f = \mathbb{R} \setminus \{\pm 1\}$

$$f'(x) = \frac{3x^2(x^2-1) - x^3 \cdot 2x}{(x^2-1)^2} = \frac{x^2(3x^2 - 3 - 2x^2)}{(x^2-1)^2} =$$

$$= \frac{x^2(x^2-3)}{(x^2-1)^2} = \frac{x^2(x-\sqrt{3})(x+\sqrt{3})}{(x^2-1)^2} = \frac{x^4 - 3x^2}{(x^2-1)^2}$$

$f'(x) = 0 \Rightarrow x_1 = -\sqrt{3},\ x_2 = 0,\ x_3 = \sqrt{3}$ ← Kritiska pkter.

$$f''(x) = \frac{(4x^3 - 6x)(x^2-1)^2 - x^2(x^2-3)\cdot 2(x^2-1)\cdot 2x}{(x^2-1)^4} =$$

$$= \frac{2x\cancel{(x^2-1)}\left((2x^2-3)(x^2-1) - 2x^2(x^2-3)\right)}{(x^2-1)^{\cancel{4}\,3}} =$$

$$= \frac{2x(\cancel{2x^4} - 2x^2 - 3x^2 + 3 - \cancel{2x^4} + 6x^2)}{(x^2-1)^3} = \frac{2x(x^2+3)}{(x^2-1)^3}$$

$f''(x) = 0 \Rightarrow x = 0$ eventuell inflektionspunkt

		$-\sqrt{3}$		-1		0		1		$\sqrt{3}$	
f'	$+$	0	$-$	ej det.	$-$	0	$-$	ej det.	$-$	0	$+$
f''	$-$		$-$	ej det.	$-$	0	$+$	ej det.	$+$		$+$

∴ f växande på $(-\infty, -\sqrt{3}] \cup [\sqrt{3}, \infty)$

f avtagande på $[-\sqrt{3}, \sqrt{3}] \setminus \{\pm 1\}$

f konkav på $(-\infty, 0] \setminus \{-1\}$

f konvex på $[0, \infty) \setminus \{1\}$