Analys 1, MVE535, vt2019, Övning 3.1

Ex. Finn alla punkter på kurvan $y=\frac{1}{x}$ där tangentlinjen är vinkelrät mot linjen $y=4x-3$.

Lösn.: $y=4x-3 \Rightarrow k_2=4 \Rightarrow$ Vi söker linjer med lutning $4\cdot k_1=-1 \Leftrightarrow k_1=-\frac{1}{4}$

Söker x så att: $y'(x)=-\frac{1}{4} \Leftrightarrow -\frac{1}{x^2}=-\frac{1}{4} \Leftrightarrow$

$\Leftrightarrow x^2=4 \Rightarrow x=\pm 2$

$y=(2)=\frac{1}{2}, \quad y(-2)=-\frac{1}{2}$

$\therefore (2,\frac{1}{2}), \ (-2,-\frac{1}{2})$

Ex. Beräkna, m.h.a. derivatans definition, derivatan till

$$F(x)=\frac{1}{\sqrt{1+x^2}}$$

Lösn.:

$$F'(x)=\lim_{h\to 0}\frac{F(x+h)-F(x)}{h}=\lim_{h\to 0}\frac{\frac{1}{\sqrt{1+(x+h)^2}}-\frac{1}{\sqrt{1+x^2}}}{h}=$$

$$=\lim_{h\to 0}\frac{\sqrt{1+x^2}-\sqrt{1+(x+h)^2}}{h\sqrt{1+(x+h)^2}\sqrt{1+x^2}}=$$

$$=\lim_{h\to 0}\frac{(\sqrt{1+x^2}-\sqrt{1+(x+h)^2})(\sqrt{1+x^2}+\sqrt{1+(x+h)^2})}{h\sqrt{1+(x+h)^2}\sqrt{1+x^2}(\sqrt{1+x^2}+\sqrt{1+(x+h)^2})}=$$

$$=\lim_{h\to 0}\frac{\cancel{1}+\cancel{x^2}-(\cancel{1}+\cancel{x^2}+2xh+h^2)}{h\sqrt{1+(x+h)^2}\sqrt{1+x^2}(\sqrt{1+x^2}+\sqrt{1+(x+h)^2})}=$$

$$=\lim_{h\to 0}\frac{-\cancel{h}(2x+h)}{\cancel{h}\sqrt{1+(x+h)^2}\sqrt{1+x^2}(\sqrt{1+x^2}+\sqrt{1+(x+h)^2})}=$$

$$=\frac{-2x}{\sqrt{1+x^2}\sqrt{1+x^2}(\sqrt{1+x^2}+\sqrt{1+x^2})}=\frac{-\cancel{2}x}{(1+x^2)\cancel{2}\sqrt{1+x^2}}=-\frac{x}{(1+x^2)^{3/2}}$$

Ex. Beräkna derivatan till:

(a) $\ln(\sqrt{x^2+1})$ (b) $\sqrt[3]{x^2 \ln(x)}$

(c) $\dfrac{\ln(x^2+1)}{\sqrt{x^2+1}}$

Lösn.: (a) $D\left(\ln(\sqrt{x^2+1})\right) = \dfrac{1}{\sqrt{x^2+1}} \cdot D\left((x^2+1)^{1/2}\right) =$

$$= \frac{1}{\sqrt{x^2+1}} \cdot \frac{1}{2}(x^2+1)^{-1/2} \cdot D(x^2+1) =$$

$$= \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+1} \cdot \sqrt{x^2+1}} = \frac{x}{x^2+1}$$

Alternativ lösning: $D\left(\ln((x^2+1)^{1/2})\right) =$

$$= D\left(\frac{1}{2}\ln(x^2+1)\right) = \frac{1}{2} \cdot \frac{1}{x^2+1} \cdot D(x^2+1) = \frac{x}{x^2+1}$$

(b) $D\left((x^2 \cdot \ln(x))^{1/3}\right) = \dfrac{1}{3}(x^2 \cdot \ln(x))^{-2/3} \cdot D(x^2 \cdot \ln(x)) =$

$$= \frac{2x \cdot \ln(x) + x^{\cancel{2}} \cdot \frac{1}{\cancel{x}}}{3(x^2 \cdot \ln(x))^{2/3}} = \frac{x(2\ln(x)+1)}{3(x^2 \cdot \ln(x))^{2/3}}$$

(c) $D\left(\dfrac{\ln(x^2+1)}{\sqrt{x^2+1}}\right) = \dfrac{D(\ln(x^2+1)) \cdot \sqrt{x^2+1} - \ln(x^2+1) D(\sqrt{x^2+1})}{(\sqrt{x^2+1})^2} =$

$$= \frac{\frac{2x}{x^2+1} \cdot \sqrt{x^2+1} - \ln(x^2+1) \cdot \frac{1}{\cancel{2}} \frac{1}{\sqrt{x^2+1}} \cdot \cancel{2}x}{x^2+1} \cdot \frac{x^2+1}{x^2+1} =$$

$$= \frac{2x\sqrt{x^2+1} - x\ln(x^2+1)\sqrt{x^2+1}}{(x^2+1)^2} =$$

$$= \frac{x\sqrt{x^2+1}\,(2-\ln(x^2+1))}{(x^2+1)^2} = \frac{x(2-\ln(x^2+1))}{(x^2+1)^{3/2}}$$

Ex. Beräkna ekvationen för tangentlinjen till kurvan

$$x + 2y + 1 = \frac{y^2}{x-1}$$

i punkten $(2,-1)$.

Lösn.: Derivera båda leden m.a.p. x:

$$1 + 2y' = \frac{2y\cdot y'(x-1) - y^2\cdot 1}{(x-1)^2}$$

Stoppa in $(2,-1)$: $1 + 2y' = \frac{2\cdot(-1)\cdot y'(2-1) - (-1)^2}{(2-1)^2}$

$\Leftrightarrow 1 + 2y' = -2y' - 1 \Leftrightarrow 4y' = -2 \Leftrightarrow$

$\Leftrightarrow y'\big|_{(2,-1)} = -\frac{1}{2}$

Tangent: $y = -\frac{1}{2}x + m$

Linjen går genom $(2,-1)$: $-1 = -\frac{1}{2}\cdot 2 + m \Leftrightarrow m = 0$

$\therefore\ y = -\frac{1}{2}x$

Bonus: Beräkna derivatan till: $y = \left(\frac{1}{x}\right)^{\ln(x)}$

Lösn.: $y = (x^{-1})^{\ln(x)} = x^{-\ln(x)} = e^{\ln(x^{-\ln(x)})} =$

$= e^{-\ln(x)\cdot\ln(x)} = e^{-(\ln(x))^2}$

$$\Rightarrow D(y) = e^{-(\ln(x))^2} \cdot D(-(\ln(x))^2) =$$

$$= e^{-(\ln(x))^2} \cdot \left(-2\ln(x) \cdot \frac{1}{x}\right) = -\frac{2\ln(x)}{x} \cdot \left(\frac{1}{x}\right)^{\ln(x)}$$