Partiella differentialekvationer—tidsberoende problem TM, 1993–11–04

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1. (a) Solve the initial value problem

$$\frac{\partial u}{\partial t}+(x+t)\frac{\partial u}{\partial x}=0,\ x\in\mathbf{R},\ t>0;\quad u(x,0)=v(x),\ x\in\mathbf{R},$$

by means of the method of characteristics. (b) Show that $\|u(\cdot,t)\|_2=e^{t/2}\|v\|_2$ for $t\geq 0$ by the energy method. (Assume that $u(x,t)\to 0$ sufficiently fast as $|x|\to\infty$.) Check the result by a direct calculation using the solution formula from (a).

2. Consider the initial-boundary value problem

$$\begin{aligned} &u_t-\Delta u=f(x,t), && x\in\Omega,\ t>0,\\ &u(x,t)=0, && x\in\Gamma,\ t>0,\\ &u(x,0)=v(x), && x\in\Omega, \end{aligned}$$

where Ω is a bounded convex domain in $\mathbf{R}^2$. (a) State a semidiscrete finite element method for this problem. (b) State and prove a stability estimate for the discrete solution operator. (c) State and prove an error estimate under the appropriate assumptions.

3. Consider solutions of the equation

$$u_{tt}-\Delta u=0,\qquad x\in\Omega,\ t>0,$$

with $u=0$ on Γ for $t>0$. Show that the energy $E(t)=\int_\Omega(|u_t(x,t)|^2+|\nabla u(x,t)|^2)\,dx$ is independent of t by means of (a) the energy method, (b) the spectral method.

4. Friedrichs' method for the initial value problem

$$\frac{\partial u}{\partial t}=a\frac{\partial u}{\partial x},\ x\in\mathbf{R},\ t>0;\quad u(x,0)=v(x),\ x\in\mathbf{R},$$

is

$$U^{n+1}(x)=\tfrac{1}{2}(1+a\lambda)U^n(x+h)+\tfrac{1}{2}(1-a\lambda)U^n(x-h),\ \lambda=k/h.$$

(a) Give a necessary and sufficient condition on λ for stability in $L_2(\mathbf{R})$. (A proof is required.) (b) Do the same for stability in $L_\infty(\mathbf{R})$. (c) Apply the Courant-Friedrichs-Lewy condition.

5. (a) State and prove the maximum principle for the heat equation. (b) Suppose that u satisfies the initial-boundary value problem

$$\begin{aligned} &u_t-\Delta u=f(x,t), && x\in\Omega,\ t>0,\\ &\frac{\partial u(x,t)}{\partial n}=g(x,t), && x\in\Gamma,\ t>0,\\ &u(x,0)=v(x), && x\in\Omega, \end{aligned}$$

where $\Omega\subset\mathbf{R}^d$ is a bounded domain in $\mathbf{R}^d$ with a smooth boundary Γ and $\partial u/\partial n$ is the exterior normal derivative. Assume in addition that $f(x,t)\geq 0$, $g(x,t)>0$ and $v(x)\geq 0$ for $x\in\Omega$, $t\geq 0$. Show that $u(x,t)\geq 0$ for $x\in\Omega$, $t\geq 0$. (It is sufficient to assume that $g(x,t)\geq 0$, but we have not learned the tool for proving this.)