Partiella differentialekvationer—tidsberoende problem TM, 1994–01–08

Telefon: Stig Larsson, 45 46 93, 0370–230 49
Inga hjälpmedel. Kalkylator ej tillåten.

$\|\cdot\|$ denotes the L_2 norm.

1. (a) Solve the initial value problem

$$\frac{\partial u}{\partial t} - (x+t)\frac{\partial u}{\partial x} = 0,\ x \in \mathbf{R},\ t > 0; \quad u(x,0) = v(x),\ x \in \mathbf{R},$$

by means of the method of characteristics. (b) Show that $\|u(\cdot,t)\| = e^{-t/2}\|v\|$ for $t \geq 0$ by the energy method. (Assume that $u(x,t) \to 0$ sufficiently fast as $|x| \to \infty$.) Check the result by a direct calculation using the solution formula from (a).

2. Consider the initial-boundary value problem

$$\begin{aligned} u_t - \Delta u &= f(x,t), && x \in \Omega,\ t > 0, \\ u(x,t) &= 0, && x \in \Gamma,\ t > 0, \\ u(x,0) &= v(x), && x \in \Omega, \end{aligned}$$

where $\Omega = \{x \in \mathbf{R}^2 : 0 < x_i < 1,\ i = 1,2\}$. (a) Formulate a finite difference method for this problem. (b) State and prove a stability estimate for the discrete solution operator. (c) State and prove an error estimate under the appropriate assumptions.

3. Consider the heat equation with Neumann's boundary condition:

$$\begin{aligned} u_t - \Delta u &= 0, && x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial n} &= 0, && x \in \Gamma,\ t > 0, \\ u(x,0) &= v(x), && x \in \Omega, \end{aligned}$$

where $\Omega \subset \mathbf{R}^2$ is a bounded domain with smooth boundary Γ and $\partial u/\partial n$ is the outward normal derivative. (a) Show that $\overline{u(t)} = \overline{v}$ for $t \geq 0$, where $\overline{v} = \frac{1}{\mathrm{vol}(\Omega)} \int_\Omega v(x)\,dx$ denotes the average value of v. (b) Show that $\|u(t) - \overline{v}\| \to 0$ as $t \to \infty$.

4. Consider the initial-boundary value problem for the wave equation

$$\begin{aligned} u_{tt} - \Delta u &= 0, && x \in \Omega,\ t > 0, \\ u &= 0, && x \in \Gamma,\ t > 0, \\ u(x,0) = v(x),\ u_t(x,0) &= w(x), && x \in \Omega, \end{aligned}$$

where $\Omega \subset \mathbf{R}^2$ is a bounded domain with smooth boundary Γ. Show the following estimates:

$$\begin{aligned} \|u(t)\|_{H_0^1} + \|u_t(t)\| &\leq C\big(\|v\|_{H_0^1} + \|w\|\big), \\ \|u_{tt}(t)\| = \|\Delta u(t)\| &\leq C\big(\|\Delta v\| + \|w\|_{H_0^1}\big), \\ \|u(t) - v\| &\leq Ct\big(\|v\|_{H_0^1} + \|w\|\big). \end{aligned}$$

5. Define the concept *Friedrichs system* (with dependent variables $u(x,t) \in \mathbf{R}^n$ and dependent variables $x \in \mathbf{R}^d$, $t > 0$). Prove an estimate of the solution of the pure initial value problem for a Friedrichs system with variable coefficients.