Matematik Chalmers

**TMA025 Partiella differentialekvationer fk, 1998–10–23**

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Inga hjälpmedel. Kalkylator ej tillåten. Uppgifterna är värda 10 poäng vardera.

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$\Omega$ is a bounded convex domain in $\mathbf{R}^2$ whose boundary $\Gamma$ is a polygon.

**1.** Consider the boundary value problem

$$\begin{aligned} -\nabla\cdot(a(x)\nabla u)+u &= f(x), \quad && x\in\Omega,\\ u &= 0, && x\in\Gamma, \end{aligned}$$

where $0<a_0\le a(x)\le a_1$ for all $x\in\Omega$.

(a) Give a weak formulation of this problem and show that it has a unique solution.

(b) Formulate a finite element method for this problem and state and prove an error estimate in the $H^1$-norm.

**2.** Let $u$ a solution of the initial-boundary value problem

$$\begin{aligned} u_t-\nabla\cdot(a(x)\nabla u)+u &= 0, \quad && x\in\Omega,\ t>0,\\ u &= 0, && x\in\Gamma,\ t>0,\\ u(x,0) &= v(x), && x\in\Omega, \end{aligned}$$

where $0<a_0\le a(x)\le a_1$ for all $x\in\Omega$. Show that ($\|\cdot\|=\|\cdot\|_{L_2(\Omega)}$)

$$\begin{aligned} \|u(t)\| &\le \|v\|, \quad t\ge 0,\\ \|\nabla u(t)\| &\le C\|\nabla v\|, \quad t\ge 0,\\ \|\nabla u(t)\| &\le Ct^{-1/2}\|v\|, \quad t>0. \end{aligned}$$

**3.** (a) Show that: if $u\in C^2(\bar\Omega)$ and $-\Delta u\le 0$ in $\Omega$, then $\max_{\bar\Omega} u=\max_\Gamma u$.

(b) Show that: if

$$\begin{aligned} -\Delta u &= f, \quad && \text{in } \Omega,\\ u &= g, && \text{on } \Gamma, \end{aligned}$$

then

$$\|u\|_{L_\infty(\Omega)}\le \|g\|_{L_\infty(\Gamma)}+C\|f\|_{L_\infty(\Omega)}.$$

**4.** (a) Write the initial value problem for the wave equation

$$\begin{aligned} &\frac{\partial^2 y}{\partial t^2}=\frac{\partial^2 y}{\partial x_1^2}+\frac{\partial^2 y}{\partial x_2^2}, \quad x\in\mathbf{R}^2,\ t>0,\\ &y(x,0)=v(x),\ y_t(x,0)=w(x), \quad x\in\mathbf{R}^2, \end{aligned}$$

as an initial value problem for a symmetric hyperbolic system (Friedrichs system). Hint: $u_1=y_t$, $u_2=y_{x_1}$, $u_3=y_{x_2}$.

(b) Prove a stability estimate for this system.

(c) Formulate Friedrichs' scheme for this system.

Vänd!

**5.** Consider the problem

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\begin{aligned}
&\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = f(x,t), \quad x \in (0,1),\ t \in (0,T),\\
&u(0,t) = g_1(t), \quad t \in [0,T],\\
&u(x,0) = g_2(x), \quad x \in [0,1],
\end{aligned}
$$

where $a > 0$ is constant and $f$, $g_1$, $g_2$ are given functions.

(a) Prove that (with $\Omega_T = (0,1) \times (0,T)$)

$$\|u\|_{L_\infty(\Omega_T)} \le \|g_1\|_{L_\infty([0,T])} + \|g_2\|_{L_\infty([0,1])} + T\|f\|_{L_\infty(\Omega_T)}.$$

Hint: the method of characteristics.

(b) Show that the difference scheme ($x_j = jh$, $t_n = nk$)

$$
\begin{aligned}
&\bar\partial_t U_j^n + a\bar\partial_x U_j^n = f(x_j,t_n), \quad j > 0,\ n > 0,\\
&U_0^n = g_1(t_n), \quad n \ge 0,\\
&U_j^0 = g_2(x_j), \quad j \ge 0,
\end{aligned}
$$

has a similar stability property for all $\lambda = k/h$.

(c) State and prove an error estimate for this difference scheme.

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