TMA025 Partiella differentialekvationer fk, 1998–10–23. Lösningar.

Ω is a bounded convex domain in $\mathbf{R}^2$ whose boundary Γ is a polygon and $\|\cdot\| = \|\cdot\|_{L_2(\Omega)}$.

1. (a) The weak formulation is

$$\begin{aligned} &u \in H_0^1(\Omega), \\ &(a\nabla u, \nabla v) + (u, v) = (f, v), \quad \forall v \in H_0^1(\Omega). \end{aligned} \tag{1}$$

We define

$$\begin{aligned} &V = H_0^1(\Omega), \quad \text{with norm } \|v\|_V = \|v\|_{H^1} = \sqrt{\|v\|^2 + \|\nabla v\|^2}, \\ &a(v, w) = (a\nabla v, \nabla w) + (v, w) = \int_\Omega a\nabla v \cdot \nabla w \, dx + \int_\Omega vw \, dx, \\ &L(v) = (f, v). \end{aligned}$$

Then the weak formulation becomes

$$\begin{aligned} &u \in V, \\ &a(u, v) = L(v), \quad \forall v \in V. \end{aligned} \tag{2}$$

We check the assumptions in Lax-Milgram's lemma. We use the assumption $0 < a_0 \le a(x) \le a_1$ for all $x \in \Omega$. The symmetric bilinear form $a(\cdot, \cdot)$ is V-elliptic:

$$\begin{aligned} a(v, v) &= (a\nabla v, \nabla v) + (v, v) \\ &\ge a_0\|\nabla v\|^2 + \|v\|^2 \\ &\ge \min(a_0, 1)\big(\|\nabla v\|^2 + \|v\|^2\big) \\ &= c_0\|v\|_{H^1}^2. \end{aligned} \tag{3}$$

$a(\cdot, \cdot)$ is bounded on V:

$$\begin{aligned} |a(v, w)| &= (a\nabla v, \nabla w) + (v, w) \\ &\le a_1\|\nabla v\| \, \|\nabla w\| + \|v\| \, \|w\| \\ &\le \max(a_1, 1)\big(\|\nabla v\| \, \|\nabla w\| + \|v\| \, \|w\|\big) \\ &\le c_1\|v\|_{H^1}\|w\|_{H^1}. \end{aligned} \tag{4}$$

The linear form $L(\cdot)$ is bounded on V:

$$|L(v)| \le \|f\| \, \|v\| \le \|f\| \, \|v\|_{H^1}.$$

Lax-Milgram's lemma now implies that (1) has a unique solution.

(b) Let $\mathcal{T}_h = \{K\}$ be a triangulation of Ω and

$$S_h = \{v \in C(\bar{\Omega}) : v|_K \text{ is linear for all } K \in \mathcal{T}_h, \, v|_\Gamma = 0\}.$$

The finite element problem is:

$$\begin{aligned} &u_h \in S_h, \\ &a(u_h, v) = L(v), \quad \forall v \in S_h. \end{aligned} \tag{5}$$

Let $e = u_h - u$. Forming the difference between (2) and (5) we get

$$a(e, v) = 0, \quad \forall v \in S_h.$$

Hence, with the standard interpolant $\pi_h u$,

$$\begin{aligned} c_0\|e\|_{H^1}^2 \le a(e, e) = a(e, u_h - u) = a(e, \pi_h u - u) \le c_1\|e\|_{H^1}\|\pi_h u - u\|_{H^1}, \\ \|e\|_{H^1} \le \frac{c_1}{c_0}\|\pi_h u - u\|_{H^1} \le \frac{c_1}{c_0} c_2 h\|u\|_{H^2}. \end{aligned}$$

We conclude that

$$\|u_h - u\|_{H^1} \le ch\|u\|_{H^2}.$$

2. The weak formulation is

$$u(t) \in H_0^1(\Omega),\ u(0) = v,$$
$$(u_t, \varphi) + a(u, \varphi) = 0, \quad \forall \varphi \in H_0^1(\Omega),$$

where $a(\cdot,\cdot)$ is the same as in Problem 1.

(a) Take $\varphi = u$:

$$(u_t, u) + a(u, u) = 0,$$
$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + a(u, u) = 0,$$
$$\|u(t)\|^2 + 2\int_0^t a(u, u)\, ds = \|v\|^2, \tag{6}$$
$$\|u(t)\| \le \|v\|.$$

(b) Take $\varphi = u_t$:

$$\|u_t\|^2 + a(u, u_t) = 0,$$
$$\|u_t\|^2 + \frac{1}{2}\frac{d}{dt}a(u, u) = 0, \qquad \text{[because } a(\cdot,\cdot) \text{ is symmetric]} \tag{7}$$
$$2\int_0^t \|u_t\|^2\, ds + a(u(t), u(t)) = a(v, v),$$
$$a(u(t), u(t)) \le a(v, v),$$
$$a_0\|\nabla u(t)\|^2 \le a(u(t), u(t)) \le a(v, v) \tag{8}$$
$$\le a_1\|\nabla v\|^2 + \|v\|^2 \le (a_1 + c)\|\nabla v\|^2, \quad \text{[by (3), (4) and Poincaré's inequality]}$$
$$\|\nabla u(t)\| \le C\|\nabla v\|.$$

(c) Multliply (7) by t:

$$t\|u_t\|^2 + \frac{1}{2}t\frac{d}{dt}a(u, u) = 0,$$
$$2t\|u_t\|^2 + \frac{d}{dt}\big(ta(u, u)\big) = a(u, u),$$
$$2\int_0^t s\|u_t\|^2\, ds + ta(u(t), u(t)) = \int_0^t a(u, u)\, ds,$$
$$a_0 t\|\nabla u(t)\|^2 \le ta(u(t), u(t)) \le \int_0^t a(u, u)\, ds \le \frac{1}{2}\|v\|^2, \quad \text{[by (6)]}$$
$$\|\nabla u(t)\| \le Ct^{-1/2}\|v\|.$$

3. See the lecture notes.

4. (a) Set $u_1 = y_t$, $u_2 = y_{x_1}$, $u_3 = y_{x_2}$. Then

$$u_{1,t} = y_{tt} = y_{x_1x_1} + y_{x_2x_2} = u_{2,x_1} + u_{3,x_2}$$
$$u_{2,t} = y_{x_1t} = u_{1,x_1}$$
$$u_{3,t} = y_{x_2t} = u_{1,x_2}$$

so that the initial value problem becomes

$$\begin{bmatrix} u_{1,t} \\ u_{2,t} \\ u_{3,t} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_{1,x_1} \\ u_{2,x_1} \\ u_{3,x_1} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_{1,x_2} \\ u_{2,x_2} \\ u_{3,x_2} \end{bmatrix}, \quad x \in \mathbf{R}^2,\ t > 0; \quad \begin{bmatrix} u_1(x,0) \\ u_2(x,0) \\ u_3(x,0) \end{bmatrix} = \begin{bmatrix} w(x) \\ v_{x_1}(x) \\ v_{x_2}(x) \end{bmatrix}.$$

This is of the form

$$\frac{\partial u}{\partial t} = A_1\frac{\partial u}{\partial x_1} + A_2\frac{\partial u}{\partial x_2}, \quad x \in \mathbf{R}^2,\ t > 0; \quad u(x,0) = u_0(x),$$

with symmetric matrices A_1, A_2, i.e., it is a symmetric hyperbolic system (= Friedrichs system).
(b) Multiply by u:

$$(u_t, u) = (A_1 u_{x_1}, u) + (A_2 u_{x_2}, u).$$

If we assume that $u = 0$ at ∞, then by integration by parts using the symmetry of A_j we get

$$(A_j u_{x_j}, u) = -(A_j u, u_{x_j}) = -(A_j u_{x_j}, u),$$

so that $(A_j u_{x_j}, u) = 0$. Hence

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 = 0$$
$$\|u(t)\| = \|u(0)\|.$$

This means that $E(t) = E(0)$, where the energy $E(t)$ is defined by

$$E(t) = \frac{1}{2}\|u(t)\|^2 = \frac{1}{2}\int_{\mathbf{R}^2} \left(y_t^2 + y_{x_1}^2 + y_{x_2}^2\right) dx.$$

(c) Friedrichs' scheme is

$$U_{i,j}^{n+1} = \tfrac{1}{2}(\tfrac{1}{2} + \lambda A_1)U_{i+1,j}^n + \tfrac{1}{2}(\tfrac{1}{2} - \lambda A_1)U_{i-1,j}^n + \tfrac{1}{2}(\tfrac{1}{2} + \lambda A_2)U_{i,j+1}^n + \tfrac{1}{2}(\tfrac{1}{2} - \lambda A_2)U_{i,j-1}^n.$$

5. (a) The characteristic curves are straight lines $x = at + c$. The curve that goes through $(\bar{x}, \bar{t})$ is

$$x = \bar{x} + a(t - \bar{t}).$$

Set $w(t) = u(\bar{x} + a(t - \bar{t}), t)$. Then

$$\frac{d}{dt}w(t) = u_t + au_x = f(\bar{x} + a(t - \bar{t}), t),$$

so that

$$w(\bar{t}) = w(t_0) + \int_{t_0}^{\bar{t}} f(\bar{x} + a(t - \bar{t}), t)\, dt$$

and

$$u(\bar{x}, \bar{t}) = \begin{cases} g_1(\bar{t} - \bar{x}/a) + \displaystyle\int_{\bar{t}-\bar{x}/a}^{\bar{t}} f(\bar{x} + a(t - \bar{t}), t)\, dt, & \bar{x} \le a\bar{t}, \\ g_2(\bar{x} - a\bar{t}) + \displaystyle\int_0^{\bar{t}} f(\bar{x} + a(t - \bar{t}), t)\, dt, & \bar{x} \ge a\bar{t}. \end{cases}$$

Hence, if $(\bar{x}, \bar{t}) \in \Omega_T$, then

$$|u(\bar{x}, \bar{t})| \le \begin{cases} \|g_1\|_{L_\infty([0,T])} + T\|f\|_{L_\infty(\Omega_T)}, & \bar{x} \le a\bar{t}, \\ \|g_2\|_{L_\infty([0,1])} + T\|f\|_{L_\infty(\Omega_T)}, & \bar{x} \ge a\bar{t}. \end{cases}$$

This implies

$$\|u\|_{L_\infty(\Omega_T)} \le \|g_1\|_{L_\infty([0,T])} + \|g_2\|_{L_\infty([0,1])} + T\|f\|_{L_\infty(\Omega_T)}.$$

(b) The scheme is

$$(1 + a\lambda)U_j^n = U_j^{n-1} + a\lambda U_{j-1}^n + kf_j^n.$$

We may write $U_j^n = V_j^n + W_j^n$, where V_j^n, W_j^n satisfy the same difference equation as U_j^n, but where

$$V_j^n \text{ has data } g_1 = 0,\ g_2 \neq 0,\ f \neq 0,$$
$$W_j^n \text{ has data } g_1 \neq 0,\ g_2 = 0,\ f = 0.$$

We begin by estimating V_j^n. Let $j_0 > 0$ be such that

$$\max_{0<j\le J} |V_j^n| = |V_{j_0}^n|. \qquad (Jh = 1)$$

Then

$$\begin{aligned}\max_{0<j\le J} |V_j^n| = |V_{j_0}^n| &= \Big|\frac{V_{j_0}^{n-1} + a\lambda V_{j_0-1}^n + kf_{j_0}^n}{1+a\lambda}\Big| \\ &\le \frac{|V_{j_0}^{n-1}| + a\lambda|V_{j_0-1}^n| + k|f_{j_0}^n|}{1+a\lambda} \\ &\le \frac{1}{1+a\lambda}\max_{0<j\le J}|V_j^{n-1}| + \frac{a\lambda}{1+a\lambda}\max_{0<j\le J}|V_j^n| + \frac{k}{1+a\lambda}\max_{j,m}|f_j^m|,\end{aligned}$$

where we used the fact that, for $n > 0$,

$$|V_{j_0-1}^n| = \left\{\begin{array}{ll}\max\limits_{0<j\le J}|V_j^n|, & j_0 \ge 2\\ 0, & j_0 = 1\end{array}\right\} \le \max_{0<j\le J}|V_j^n|.$$

Hence, for $0 < n \le N$,

$$\begin{aligned}\max_{0<j\le J}|V_j^n| &\le \max_{0<j\le J}|V_j^{n-1}| + k\max_{j,m}|f_j^m| \\ &\le \cdots \le \max_{0\le j\le J}|V_j^0| + nk\max_{j,m}|f_j^m| \qquad [\text{note } j=0 \text{ here}] \\ &\le \max_{0\le j\le J}|g_2(x_j)| + T\max_{j,m}|f_j^m|. \qquad (nk \le Nk = T)\end{aligned}$$

In order to estimate W_j^n we interchange the roles of j and n. Let $n_0 > 0$ be such that

$$\max_{0<n\le N}|W_j^n| = |W_j^{n_0}|. \qquad (Nk = T)$$

Then

$$\begin{aligned}\max_{0<n\le N}|W_j^n| = |W_j^{n_0}| &= \Big|\frac{W_j^{n_0-1} + a\lambda W_{j-1}^{n_0}}{1+a\lambda}\Big| \\ &\le \frac{|W_j^{n_0-1}| + a\lambda|W_{j-1}^{n_0}|}{1+a\lambda} \\ &\le \frac{1}{1+a\lambda}\max_{0<n\le N}|W_j^n| + \frac{a\lambda}{1+a\lambda}\max_{0<n\le N}|W_{j-1}^n|,\end{aligned}$$

where we used the fact that, for $j > 0$,

$$|W_j^{n_0-1}| = \left\{\begin{array}{ll}\max\limits_{0<n\le N}|W_j^n|, & n_0 \ge 2\\ 0, & n_0 = 1\end{array}\right\} \le \max_{0<n\le N}|W_j^n|.$$

Hence, for $0 < j \le J$,

$$\begin{aligned}\max_{0<n\le N}|W_j^n| &\le \max_{0<n\le N}|W_{j-1}^n| \\ &\le \cdots \le \max_{0\le n\le N}|W_0^n| \qquad [\text{note } n=0 \text{ here}] \\ &= \max_{0\le n\le N}|g_1(t_n)|.\end{aligned}$$

We conclude

$$\begin{aligned}\max_{j,n}|U_j^n| &\le \max_{j,n}|V_j^n| + \max_{j,n}|W_j^n| \\ &\le \max_{0\le n\le N}|g_1(t_n)| + \max_{0\le j\le J}|g_2(x_j)| + T\max_{j,n}|f_j^n|.\end{aligned}$$

Here $\max_{j,n}$ means the maximum with respect to $0 < j \le J$, $0 < n \le N$.

(c) Set $z_j^n = U_j^n - u_j^n$. Then

$$\begin{aligned}
&\bar\partial_t z_j^n + a\bar\partial_x z_j^n = -\tau_j^n, \quad j > 0,\ n > 0,\\
&z_0^n = 0, \quad n \ge 0,\\
&z_j^0 = 0, \quad j \ge 0,
\end{aligned}$$

where the local truncation error is

$$\tau_j^n = \bar\partial_t u_j^n + a\bar\partial_x u_j^n - f_j^n.$$

The stability estimate from (b) and Taylor's formula give

$$\begin{aligned}
\max_{j,n} |z_j^n| &\le T \max_{j,n} |\tau_j^n|\\
&\le TC(u)(h+k) = C(u,T)h \qquad [\text{because } k = \lambda h].
\end{aligned}$$

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