Matematik Chalmers

TMA026/MMA430 Partial differential equations II
Partiella differentialekvationer II, 2014–08–29 f V

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Inga hjälpmedel. Kalkylator ej tillåten. No aids or electronic calculators are permitted.

You may get up to 10 points for each problem plus points for the hand-in problems.
Grades: 3: 20–29p, 4: 30–39p, 5: 40–.

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1. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain, with boundary Γ. Consider the diffusion-reaction problem,

$$\begin{cases} -\Delta u + cu = f, & \text{in } \Omega, \\ u = 0, & \text{on } \Gamma, \end{cases}$$

with $f \in L^2(\Omega)$ and $0 \le c \le \beta < \infty$.

(a) Derive the weak form.
(b) Formulate the finite element method and derive an error estimate in energy norm $\|\nabla(u - u_h)\|_{L^2(\Omega)}$.
(c) Is there a negative constant $c < 0$ such that the bilinear form $a(u, v) = (\nabla v, \nabla v) + (cv, v)$ is still coercive? (Motivate your answer)

2. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain, with boundary Γ, and $I = (0, T)$. Consider the initial value problem,

$$\begin{cases} \dot{u} - \Delta u + \gamma u = 0, & \text{in } \Omega \times I, \\ u = 0, & \text{on } \Gamma \times I, \\ u(\cdot, 0) = v, & \text{in } \Omega, \end{cases}$$

with $v \in L^2(\Omega)$ and $\gamma \in \mathbb{R}^+$.

(a) Express the solution in terms of the eigenfunctions and eigenvalues of $-\Delta$.
(b) How does the $L^2(\Omega)$ norm of the solution for a fixed time t depend on the parameter γ.
(c) Show that $\int_0^t \|\dot{u}(s)\|^2_{L^2(\Omega)}\, ds$ is bounded for all $t \ge 0$.

3. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain, with boundary Γ, and let $I = (0, T)$. Consider the semi-linear parabolic problem,

$$\begin{cases} \dot{u} - \Delta u = f(u) := u - u^3, & \text{in } \Omega \times I, \\ u = 0, & \text{on } \Gamma \times I, \\ u(\cdot, 0) = v, & \text{in } \Omega. \end{cases} \tag{1}$$

Assume $\|v\|_{H^1(\Omega)} \le R_0$.

(a) Let $\|u(t)\|_{H^1(\Omega)} \le R$ and $\|w(t)\|_{H^1(\Omega)} \le R$ for $0 \le t \le T$. Show that,

$$\|f(u) - f(w)\|_{L^2(\Omega)} \le C(R)\|u - w\|_{H^1(\Omega)}, \quad t \in [0, T].$$

Hint: The inequality $\|w\|_{L^p(\Omega)} \le C\|w\|_{H^1(\Omega)}$ *holds for* $1 \le p \le 6$.

(b) The solution to equation (2) can be written using the parabolic solution operator $E(t)$ (the solution operator to equation (3)), in the following way,

$$u(t) = E(t)v + \int_0^t E(t - s) f(u(s))\, ds.$$

Let $Su(t) = E(t)v + \int_0^t E(t - s) f(u(s))\, ds$, $\mathcal{B} = \{w : \max_{0 \le t \le \tau} \|w(t)\|_{H^1(\Omega)} \le R\}$, and show that $S : \mathcal{B} \to \mathcal{B}$ for sufficiently large R and small τ.

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(c) Show that S is also a contraction mapping (and therefore has a fixed point $u = Su$), i.e. show

$$\max_{t\in[0,\tau]} \|Su - Sw\|_{H^1(\Omega)} \leq \gamma \max_{t\in[0,\tau]} \|u - w\|_{H^1(\Omega)},$$

where $\gamma < 1$, for sufficiently small τ.

4. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain, with boundary Γ, and $I = (0, T)$. Consider the initial value problem,

$$\begin{cases} \dot{u} - \Delta u = 0, & \text{in } \Omega \times I, \\ u = 0, & \text{on } \Gamma \times I, \\ u(\cdot, 0) = v, & \text{in } \Omega, \end{cases} \tag{2}$$

with $v \in L^2(\Omega)$.

(a) Formulate the Galerkin finite element method with Backward Euler time-stepping for the problem.
(b) Show that the $L^2(\Omega)$ norm of the solution is bounded by the initial value for all $t \geq 0$.
(c) Assume we have a problem with a smooth solution for all times discretized using FEM-Backward Euler with continuous piecewise linear basis functions. Further assume we can evaluate the error in $L^2(\Omega)$ norm for a fixed time t. How will the error change with the time-step k and the mesh parameter h respectively?

5. Let $\Omega = [0, 1]^2$ be the unit square. Show that for all $v \in C^1(\bar{\Omega})$,

$$\|v\|_{L^2(\partial\Omega)} \leq C\|v\|_{H^1(\Omega)}.$$

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