Matematik Chalmers

TMA026/MMA430 Partial differential equations II
Partiella differentialekvationer II, 2015–06–02 f V

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Inga hjälpmedel. Kalkylator ej tillåten. No aids or electronic calculators are permitted.

You may get up to 10 points for each problem plus points for the hand-in problems.
Grades: 3: 20p–29p, 4: 30p–39p, 5: 40p–, G: 20p–34p, VG: 35p–

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1. Consider the following problems.

(a) Compute the weak derivative of $|x+2|$ on the interval $[-4,4]$.
(b) Let $\Omega = \{x \in \mathbb{R}^2 : |x| \le 1/2\}$. Show that $v(x) = \log(-\log(|x|^2))$ belongs to $H^1(\Omega)$.
(c) Let $\Omega = [0,1]$ and $f = x$. Show $\|f\|_{H^{-1}(\Omega)} = \sup_{v \in H_0^1(\Omega)} \frac{|(f,v)|}{\|v'\|_{L^2(\Omega)}} \le \frac{1}{2\sqrt{5}}$.

2. Let $\Omega \subset \mathbb{R}^d$ be a convex domain, with boundary Γ. Consider the Poisson equation with diffusion coefficient $a \in C^1(\bar{\Omega})$, $0 < a_0 \le a \le a_1$,

$$\begin{cases} -\nabla \cdot (a \nabla u) = f, & \text{in } \Omega, \\ u = 0, & \text{on } \Gamma. \end{cases}$$

(a) Now let $\tilde{u}$ solve the same equation but with a replaced by $\tilde{a}$ (also fulfilling $a_0 \le \tilde{a} \le a_1$). Formulate the corresponding weak form (for $\tilde{u}$) and bound $\|u - \tilde{u}\|_{H^1(\Omega)}$.
(b) Formulate the finite element method with $\tilde{a}$. Let the corresponding finite element approximation be denoted $\tilde{u}_h \in V_h \subset H_0^1(\Omega)$. Derive an error bound for $\|u - \tilde{u}_h\|_{H^1(\Omega)}$.
(c) Discuss why using an inexact diffusion coefficient in the numerical scheme, e.g. $\tilde{a} = I_h a$ (an interpolant), can be useful from an application point of view.

3. Let $\Omega \subset \mathbb{R}^d$ be a convex domain, with boundary Γ. Consider the heat equation,

$$\begin{cases} \dot{u} - \Delta u = f, & \text{in } \Omega \times I, \\ u = 0, & \text{on } \Gamma \times I, \\ u(\cdot, 0) = v, & \text{in } \Omega, \end{cases}$$

where v and f are smooth and $I = (0, T)$.

(a) Formulate the Backward Euler Galerkin finite element method for this problem.
(b) Show that the $L^2(\Omega)$ norm of the Backward Euler Galerkin finite element approximation is decaying in time if $f = 0$.
(c) If the solution u is smooth, how does the error in the approximation depend on the mesh size h and time step k? How is this modified for the Crank-Nicolson-Galerkin method?

4. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain, with smooth boundary Γ. Consider the wave equation,

$$\begin{cases} \ddot{u} - \Delta u = 0, & \text{in } \Omega \times I, \\ u = 0, & \text{on } \Gamma \times I, \\ u(\cdot, 0) = v, \quad \dot{u}(\cdot, 0) = w, & \text{in } \Omega, \end{cases}$$

where v and w are smooth.

(a) Express the solution as a linear combination of the eigenfunctions of $-\Delta$.
(b) Show that the total energy of u is constant in time.

5. *The maximum principle:* Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with smooth boundary Γ. Let $u \in C^2(\bar{\Omega})$. Show that if $\Delta u \le 0$ for all $x \in \Omega$, then $\max_{\bar{\Omega}} u = \max_{\Gamma} u$.

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