Matematik Chalmers

TMA026/MMA430 Partial differential equations II
Partiella differentialekvationer II, 2016–05–31 f M

Telefon: Axel Målqvist 031-7723599
Inga hjälpmedel. Kalkylator ej tillåten. No aids or electronic calculators are permitted.

You may get up to 10 points for each problem plus points for the hand-in problems.
Grades: 3: 20p–29p, 4: 30p–39p, 5: 40p–, G: 20p–34p, VG: 35p–

---

1. Sol.

(a) We have $\phi_1(x) = \frac{x}{h}$ on $0 \leq x \leq h$, $\phi_1(x) = 2 - \frac{x}{h}$ on $h \leq x \leq 2h$, and zero otherwise. The weak derivative is therefore $D_w\phi_1 = \frac{1}{h}$ for $0 \leq x \leq h$, $D_w\phi_1 = -\frac{1}{h}$ for $h \leq x \leq 2h$, and zero otherwise since,

$$-\int_\Omega \phi_1 \frac{\partial \psi}{\partial x}\, dx = \int_\Omega D_w\phi_1 \psi\, dx, \quad \forall \psi \in C_0^1.$$

(b) Since the mesh is quasi uniform (with constant C) and shape regular (with constant C_ρ), the largest inscribed ball with in any element in $\mathcal{T}_h$ has radius greater than $C_\rho C^{-1}h$ in any element. Therefore $|\nabla\phi_i| \leq C_\rho^{-1}Ch^{-1}$ for all $x \in \Omega$, where C' is independent of h. We get $\|\nabla\phi_i\|_{L^p(\Omega)} \leq C'h^{-1}(\int_{\mathrm{supp}(\phi_i)} 1\, dx)^{1/p} = C'h^{d/p-1}$. Therefore, $\|\nabla\phi_i\|_{L^p(\Omega)} \leq C'$ independent of h, if $1 \leq p \leq d$.

2. Sol. See Theorem 3.8 in Thomée-Larsson.

3. Sol.

(a) The representation is valid if $\omega^2 \neq \lambda_i$ for any $i = 1, \ldots$. For those ω^2 we have $u = \sum_{i=1}^\infty \frac{(f,\varphi_i)}{\lambda_i - \omega^2}\varphi_i$.
(b) Let $v = \sum_{i=1}^\infty \alpha_i\varphi_i$. Then,

$$(\nabla v, \nabla v) - \omega^2(v, v) = \sum_{i=1}^\infty (\lambda_i - \omega^2)\alpha_i^2 \geq \frac{1}{2}\sum_{i=1}^\infty \lambda_i\alpha_i^2 \geq \frac{1}{2}|v|_{H^1(\Omega)}^2.$$

4. Sol.

(a) We let S_h be the space of continuous piecewise linear functions fulfilling the boundary condition. We pick a uniform time step k and let $\bar{\partial}_t U^n = \frac{U^n - U^{n-1}}{k}$. Find $U^n \in S_h$ such that,

$$(\bar{\partial}_t U^n, \chi) + (\nabla U^n, \nabla\chi) = (f(t_n), \chi), \quad \forall \chi \in S_h, \quad n \geq 1,$$

with $U^0 = v_h$.
(b) We let the test function be U^n and use that $(\nabla U^n, \nabla U^n) \geq 0$ to get $(\bar{\partial} U^n, U^n) \leq \|f(t_n)\|_{L^2(\Omega)}\|U^n\|_{L^2(\Omega)}$. We conclude,

$$\|U^n\|_{L^2(\Omega)}^2 \leq (\|U^{n-1}\|_{L^2(\Omega)} + \|f(t_n)\|_{L^2(\Omega)})\|U^n\|_{L^2(\Omega)}.$$

We divide with $\|U^n\|_{L^2(\Omega)}$ and repeat the argument to get,

$$\|U^n\|_{L^2(\Omega)} \leq \|v_h\|_{L^2(\Omega)} + k\sum_{j=1}^n \|f(t_j)\|_{L^2(\Omega)}.$$

(c) We have $\|u(t_n) - U^n\|_{L^2(\Omega)} \leq C_1h^2 + C_2k$. Therefore $k \sim h^2$ would balance the terms.

5. Sol.

(a) We have $(A_j \frac{\partial u}{\partial x_j}, u) = \frac{1}{2}\frac{\partial}{\partial x_j}(A_j u, u) - (\frac{\partial A_j}{\partial x_j} u, u) = 0$ since u vanishes for large x and A_j is constant. We multiply the equation with u and integrate in space to get,

$$\|u\|_{L^2(\mathbb{R}^d)} \frac{\partial}{\partial t}\|u\|_{L^2(\mathbb{R}^d)} = \frac{1}{2}\frac{\partial}{\partial t}\|u\|^2_{L^2(\mathbb{R}^d)} = (\dot{u}, u) = (f, u) \leq \|f\|_{L^2(\mathbb{R}^d)}\|u\|_{L^2(\mathbb{R}^d)}.$$

We divide by $\|u\|_{L^2(\mathbb{R}^d)}$ and integrate in time,

$$\|u(t)\|_{L^2(\mathbb{R}^d)} \leq \|v\|_{L^2(\mathbb{R}^d)} + \int_0^t \|f(s)\|_{L^2(\mathbb{R}^d)}\, ds.$$

(b) We have the problem $u'_t(x,t) + A_1 u'_x(x,t) = 0$ and $u(x,0) = v(x)$. We let t parameterize the problem and get $\frac{d}{dt}x(t) = A_1$ and therefore the characteristic curve $x(t) = A_1 t + C$. The characteristic through $(\bar{x}, \bar{t})$ is given by $\bar{x} = A_1\bar{t} + C$ or $C = \bar{x} - A_1\bar{t}$. We have that the solution is constant along the characteristic line. Therefore $u(\bar{x}, \bar{t}) = u(x(\bar{t}, \bar{t}) = u(x(0), 0) = u(C, 0) = v(\bar{x} - A_1\bar{t})$.

/axel